# 6

# Black-body radiation

The subjects for consideration in this chapter are the black-body model, which is of primary importance in thermal radiation theory and practice, and the fundamental laws of radiation of such a system. Natural and artificial physical objects, which are close in their characteristics to black bodies, are considered here. The quantitative black-body radiation laws and their corollaries are analysed in detail. The notions of emissivity and absorptivity of physical bodies of grey-body radiation character are also introduced. The Kirchhoff law, its various forms and corollaries are analysed on this basis.

## 6.1 THE IDEAL BLACK-BODY MODEL: HISTORICAL ASPECTS

The ideal black-body notion (hereafter the black-body notion) is of primary importance in studying thermal radiation and electromagnetic radiation energy transfer in all wavelength bands. Being an ideal radiation absorber, the black body is used as a standard with which the absorption of real bodies is compared. As we shall see later, the black body also emits the maximum amount of radiation and, consequently, it is used as a standard for comparison with the radiation of real physical bodies. This notion, introduced by G. Kirchhoff in 1860, is so important that it is actively used in studying not only the intrinsic thermal radiation of natural media, but also the radiations caused by different physical nature. Moreover, this notion and its characteristics are sometimes used in describing and studying artificial, quasi-deterministic electromagnetic radiation (in radio- and TV-broadcasting and communications). The emissive properties of a black body are determined by means of quantum theory and are confirmed by experiment.

The black body is so called because those bodies that absorb incident visible light well seem black to the human eye. The term is, certainly, purely conventional and has, basically, historical roots. For example, we can hardly characterize our Sun, which is, indeed, almost a black body within a very wide band of electromagnetic radiation wavelengths, as a black physical object in optics. Though, it is namely the bright-white sunlight, which represents the equilibrium black-body radiation. In this sense, we should treat the subjective human recognition of colours extremely cautiously. So, in the optical band a lot of surfaces really approach an ideal back body in their ability to absorb radiation (examples of such surfaces are: soot, silicon carbide, platinum and golden niellos). However, outside the visible light region, in the wavelength band of IR thermal radiation and in the radio-frequency bands, the situation is different. So, the majority of the Earth's surfaces (the water surface, ice, land) absorb infrared radiation well, and, for this reason, in the thermal IR band these physical objects are ideal black bodies. At the same time, in the radio-frequency band the absorptive properties of the same media differ both from a black body and from each other, which, generally speaking, just indicates the high information capacity of microwave remote measurements.

### 6.1.1 Definition of a black body

A black body is an ideal body which allows the whole of the incident radiation to pass into itself (without reflecting the energy) and absorbs within itself this whole incident radiation (without passing on the energy). This property is valid for radiation corresponding to all wavelengths and to all angles of incidence. Therefore, the black body is an ideal absorber of incident radiation. All other qualitative characteristics determining the behaviour of a black body follow from this definition (see, for example, Siegel and Howell, 1972; Ozisik, 1973).

### 6.1.2 Properties of a black body

A black body not only absorbs radiation ideally, but possesses other important properties which will be considered below.

Consider a black body at constant temperature, placed inside a fully insulated cavity of arbitrary shape, whose walls are also formed by ideal black bodies at constant temperature, which initially differs from the temperature of the body inside. After some time the black body and the closed cavity will have a common equilibrium temperature. Under equilibrium conditions the black body must emit exactly the same amount of radiation as it absorbs. To prove this, we shall consider what would happen if the incoming and outgoing radiation energies were not equal. In this case the temperature of a body placed inside a cavity would begin to increase or decrease, which would correspond to heat transfer from a cold to a heated body. But this situation contradicts the second law of thermodynamics (the question is, certainly, on the stationary state of an object and ambient radiation). Since, by definition, the black body absorbs a maximum possible amount of radiation that comes in any direction from a closed cavity at any wavelength, it should also emit a maximum possible amount of radiation (*as an ideal emitter*). This situation becomes clear if we consider any less perfectly absorbing body (a grey body), which should emit a lower amount of radiation as compared to the black body, in order that equilibrium be maintained.

Let us now consider an isothermal closed cavity of arbitrary shape with black walls. We move the black body inside the cavity into another position and change its orientation. The black body should keep the same temperature, since the whole closed system remains isothermal. Therefore, the black body should emit the same amount of radiation as before. Being at equilibrium, it should receive the same amount of radiation from the cavity walls. Thus, the total radiation received by the black body does not depend on its orientation and position inside the cavity; therefore, the radiation passing through any point inside a cavity does not depend on its position or on the direction of emission. This implies that the equilibrium thermal radiation filling a cavity is isotropic (the property of isotropy of black-body radiation). And, thus, the net radiation flux (see equation (5.7)) through any plane, placed inside a cavity in any arbitrary manner, will be strictly zero.

Consider now an element of the surface of a black isothermal closed cavity and the elementary black body inside this cavity. Some part of the surface element's radiation falls on a black body at some angle to its surface. All this radiation is absorbed, by definition. In order that the thermal equilibrium and radiation isotropy be kept throughout the closed cavity, the radiation emitted by a body in the direction opposite to the incident beam direction should be equal to the absorbed radiation. Since the body absorbs maximum radiation from any direction, it should also emit maximum radiation in any direction. Moreover, since the equilibrium thermal radiation filling the cavity is isotropic, the radiation absorbed or emitted in any direction by the ideal black surface encased in the closed cavity, and related to the unit area of surface projection on a plane normal to the beam direction, should be equal.

Let us consider a system comprising a black body inside a closed cavity which is at thermal equilibrium. The wall of the cavity possesses a peculiar property: it can emit and absorb radiation within a narrow wavelength band only. The black body, being an ideal energy absorber, absorbs the whole incident radiation in this wavelength band. In order that the thermal equilibrium be kept in a closed cavity, the black body should emit radiation within the aforementioned wavelength band; and this radiation can then be absorbed by the cavity wall, which absorbs in the given wavelength band only. Since the black body absorbs maximum radiation in a certain wavelength band, it should emit maximum radiation in the same band. The black body should also emit maximum radiation at the given wavelength. Thus, the black body is an ideal emitter at any wavelength. However, this in no way implies uniformity in the intensity of black-body emission at different wavelengths (the 'white noise' property). The peculiar spectral (and, accordingly, correlation) properties of black-body radiation could only be revealed by means of quantum mechanics.

The peculiar properties of a closed cavity have no relation to the black body in the reasoning given, since the emission properties of a body depend on its nature only and do not depend on the properties of a cavity. The walls of a cavity can even be fully reflecting (mirroring).

If the temperature of a closed cavity changes, then, accordingly, the temperature of a black body enclosed inside it should also change and become equal to the new temperature of a cavity (i.e. a fully insulated system should tend to thermodynamic equilibrium). The system will again become isothermal, and the energy of radiation absorbed by a black body will again be equal to the energy of radiation emitted by it, but it will slightly differ in magnitude from the energy corresponding to the former temperature. Since, by definition, the body absorbs (and, hence, emits) the maximum radiation corresponding to the given temperature, the characteristics of an enclosing system have no influence on the emission properties of a black body. Therefore, the total radiation energy of a black body is a function of its temperature only.

In addition, according to the second law of thermodynamics, energy transfer from a cold surface to a hot one is impossible without doing some work at a system. If the energy of radiation emitted by a black body increased with decreasing temperature, then the reasoning could easily be constructed (see, for example, Siegel and Howell, 1972), which would lead us to a violation of this law. As an example, two infinite parallel ideal black plates are usually considered. The upper plate is maintained at temperature higher than the temperature of the lower plate. If the energy of emitted radiation decreased with increasing temperature, then the energy of radiation, emitted by the lower plate per unit time, would be greater than the energy of radiation emitted by the upper plate per unit time. Since both plates are black, each of them absorbs the whole radiation emitted by the other plate. For maintaining the temperatures of plates the energy should be rejected from the upper plate per unit time and added to the lower plate in the same amount. Thus, it happens, that the energy transfers from a less heated plate to more heated one without any external work being done. According to the second law of thermodynamics, this situation is impossible. Therefore, the energy of radiation emitted by a black body, should increase with temperature. On the basis of these considerations we come to the conclusion, that the total energy of radiation emitted by a black body is proportional to a monotonously increasing function of thermodynamic temperature only.

All the reasoning we set forth above proceeding from thermodynamic considerations represents quite important, but, nevertheless, only qualitative, laws of black-body radiation. As was ascertained, classical thermodynamics is not capable of formulating the quantitative laws of black-body radiation in principle.

### 6.1.3 Historical aspects

Until the middle of the nineteenth century a great volume of diverse experimental data on the radiation of heated bodies was accumulated. The time had come to comprehend the data theoretically. And it was Kirchhoff who took two important steps in this direction. At the first step Kirchhoff, together with Bunsen, established the fact that a quite specific spectrum (the set of wavelengths, or frequencies) of the light emitted and absorbed by a substance corresponds to that particular substance. This discovery served as a basis for the spectral analysis of substances. The second step consisted in finding the conditions, under which the radiation spectrum of

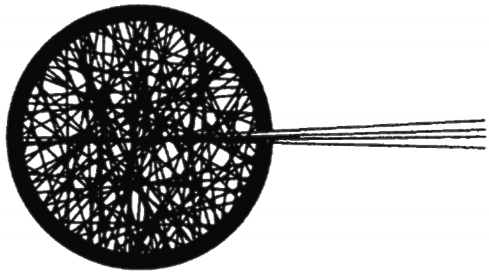

**Figure 6.1.** Classic experimental model of black-body source.

heated bodies depends only on their temperature and does not depend on the chemical composition of the emitting substance. Kirchhoff considered theoretically the radiation inside a closed cavity in a rigid body, whose walls possess some particular temperature. In such a cavity the walls emit as much energy as they absorb. It was found that under these conditions the energy distribution in the radiation spectrum does not depend on the material the walls are made of. Such a radiation was called 'absolutely (or ideally) black'.

For a long time, however, black-body radiation was, so to speak, a 'thing-in-itself'. Only 35 years later, in 1895, W. Wien and O. Lummer suggested the development of a test model of an ideal black body to verify Kirchhoff's theory experimentally. This model was manufactured as a hollow sphere with internal reflecting walls and a narrow hole in the wall, the hole diameter being small as compared to the sphere diameter. The authors proposed to investigate the spectrum of radiation issuing through this hole (Figure 6.1). Any light beam undergoes multiple reflections inside a cavity and, actually, cannot exit through the hole. At the same time, if the walls are at a high temperature the hole will brightly shine (if the process occurs in the optical band) owing to the electromagnetic radiation issuing from inside the cavity. It was this particular test model of a black body on which the experimental investigations to verify thermal radiation laws were carried out, and, first of all, the fundamental spectral dependence of black-body radiation on frequency and temperature (the Planck formula) was established quantitatively. The success of these experimental (and, a little bit later, theoretical) quantum-approach-based investigations was so significant that for a long time, up until now, this famous reflecting cavity has been considered in general physics textbooks as a unique black-body example. And, thus, some illusion of black body exclusiveness with respect to natural objects arises. In reality, however (as we well know both from the radio-astronomical and remote sensing data, and from the data of physical (laboratory) experiments), the natural world around us, is virtually saturated with physical objects which are very close to black-body models in their characteristics.

First of all, we should mention here *the cosmic microwave background (CMB) of the universe* – the fluctuation electromagnetic radiation that fills the part of the universe known to us. The radiation possesses nearly isotropic spatial-angular field with an intensity that can be characterized by the radiobrightness temperature of 2.73 K. The microwave background is, in essence, some kind of 'absolute ether at rest' that physicists intensively sought at the beginning of the twentieth century. A small dipole component in the spatial-angular field of the microwave background allowed the researchers to determine, to a surprising accuracy, the direction and velocity of motion of the solar system. The contribution of the microwave background as a re-reflected radiation should certainly be taken into account in performing fine investigations of the emissive characteristics of terrestrial surfaces from spacecraft.

The second (but not less important) source of black-body radiation is the star nearest to the Earth – *the Sun* (see section 1.4). The direct radar experiments, performed in the 1950s and 1960s, have indicated a complete absence of a radio-echo (within the limits of the receiving equipment capability) within the wide wavelength band – in centimetre, millimetre and decimetre ranges. Detailed spectral studies of solar radiation in the optical and IR bands have indicated the presence of thermal black-body radiation with a brightness temperature of 5800 K at the Sun. In other bands of the electromagnetic field the situation is essentially more complicated – along with black-body radiation there exist powerful, non-stationary quasi-noise radiations (flares, storms), which are described, nevertheless, in thermal radiation terms.

The third space object is our home planet, – *the Earth*, which possesses radiation close to black-body radiation with a thermodynamic temperature of 287 K. The basic radiation energy is concentrated in the 8–12 micrometre band, in which almost all terrestrial surfaces possess black-body radiation properties. Just that small portion of radiation energy which falls in the radio-frequency band is of interest for microwave sensing. The detailed characteristics of radiation from terrestrial surfaces in this band have shown serious distinctions of many terrestrial media from the black-body model.

In experimental measurements of the radiation properties of real physical bodies it is necessary to have an ideally black surface or a black emitter as a standard. Since ideal black sources do not exist, some special technological approaches are applied to develop a realistic black-body model. So, in optics these models represent hollow metal cylinders having a small orifice and cone at the end, which are immersed in a thermostat with fixed (or reconstructed) temperature (Siegel and Howell, 1972). In the radio-frequency band segments of waveguides or coaxial lines, filled with absorbing substance (such as carbon-containing fillers), are applied. Multilayer absorbing covers, which are widely used in the military-technological area (for instance, Stealth technology), are applied as standard black surfaces in this band. It is clear, that objects covered with such an absorbing coat are strong emitters of the fluctuation electromagnetic field. It is important also to note that in the radio-frequency band a closed space with well-absorbing walls (such as a concrete with various fillers) represents a black-body cavity to a good approximation. For these reasons the performance of fine radiothermal investigations in closed rooms (indoors) makes no sense. (Of interest is the fact that it was in a closed laboratory room that in 1888 Hertz managed to measure for the first time the wavelength of electromagnetic radiation.)

## 6.2 BLACK-BODY RADIATION LAWS

But now we return to the quantitative laws of black-body radiation. The general thermodynamic considerations allowed Kirchhoff, Boltzmann and Wien to derive rigorously a series of important laws controlling the emission of heated bodies. However, these general considerations were insufficient for deriving a particular law of energy distribution in the ideal black-body radiation spectrum. It was W. Wien who advanced in this direction more than the others. In 1893 he spread the notions of temperature and entropy to thermal radiation and showed, that the maximum radiation in the black-body spectrum displaces to the side of shorter wavelengths with increasing temperature (the Wien displacement law); and at a given frequency the radiation intensity can depend on temperature only, as the parameter appeared in the $(\nu/T)$ ratio. In other words, the spectral intensity should depend on some function $f(\nu/T)$. The particular form of this function has remained unknown.

In 1896, proceeding from classical concepts, Wien derived the law of energy distribution in the black-body spectrum (the Wien radiation law). However, as was soon made clear, the formula of Wien's radiation law was correct only in the case of short (in relation to the intensity maximum) waves. Nevertheless, these two laws of Wien have played a considerable part in the development of quantum theory (the Nobel Prize, 1911).

J. Rayleigh (1900) and J. Jeans (1905) derived the spectral distribution of thermal radiation on the basis of the assumption that the classical idea on the uniform distribution of energy is valid. However, the temperature and frequency dependencies obtained basically differed from Wien's relationships.

According to the results of fairly accurate measurements, carried out before that time, and to some theoretical investigations, Wien's expression for spectral energy distribution was invalid at high temperatures and long wavelengths. This circumstance forced Planck to turn to consideration of harmonic oscillators, which have been taken as the sources and absorbers of radiation energy. Using some further assumptions on the mean energy of oscillators, Planck derived Wien's and the Rayleigh–Jeans' laws of radiation. Finally, Planck obtained the empirical equation, which very soon was reliably confirmed experimentally on the basis, first of all, of the Wien–Lummer black-body model. Searching for the theory modifications which would allow this empirical equation to be derived, Planck arrived at the assumptions constituting the quantum theory basis (the Nobel Prize, 1918).

### 6.2.1 The Planck law (formula)

According to quantum statistics principles, the spectral volume density of radiation energy can be determined (see relation (5.10)) by calculating the equilibrium distribution of photons, for which the radiation field entropy is maximum, and taking into consideration that the photon energy with frequency $\nu$ is equal to $h\nu$, where $h$ is the Planck constant (Table A.4). If the radiation field is considered to be a gas obeying the Einstein–Bose statistics, then we obtain the Planck formula for the volume density of radiation (see, for example, Schilling, 1972; Amit and Verbin, 1999):

$$u_\nu(T)\,\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{[\exp(h\nu/kT) - 1]}\,\mathrm{d}\nu, \tag{6.1}$$

where $k$ is the Boltzmann constant (Table A.4).

Apart from a rigorous quantum derivation of Planck's formula, there exists a spectrum of heuristic approaches (see, for example, Penner, 1959).

From the remote sensing point of view, of principal significance is the other radiation field characteristic, namely, the spectral radiation intensity, which is measured at once by remote sensing devices. With allowance for relation (5.12), the spectral intensity of black-body radiation into the transparent medium with refractive index $n$ will be specified by the following expression:

$$I_\nu(T,\nu) = \frac{2h\nu^3 n^2}{c_0^2}\frac{1}{[\exp(h\nu/kT) - 1]}. \tag{6.2}$$

It can easily be seen from this relation that the black-body radiation into the transparent medium is $n^2$ times greater than when emitting into a vacuum (the Clausius law).

In many practical applications in determining the spectral intensity of radiation the wavelength is used instead of frequency. It is impossible to transfer from frequency to wavelength by simply replacing the frequency with the wavelength in expression (6.2), because this expression includes the differential quantity. However, this expression can be transformed taking into account that the energy of radiation, emitted within the frequency band $\mathrm{d}\nu$, that includes frequency $\nu$, is equal to the energy of radiation, emitted within the wavelength band $\mathrm{d}\lambda$ that includes the working wavelength $\lambda$,

$$I_\nu(T,\nu)|\mathrm{d}\nu| = I_\lambda(T,\lambda)|\mathrm{d}\lambda|. \tag{6.3}$$

The wavelength depends on the medium, in which the radiation propagates (see section 1.6). Subscript 0 denotes that the considered medium is the vacuum. At the same time, the electromagnetic radiation frequency does not depend on the medium. The frequency and wavelength in a transparent dielectric medium ($\lambda$) are related by the equation:

$$\nu = \frac{c_0}{n\lambda}. \tag{6.4}$$

Supposing the refractive index of a transparent medium to be independent of the frequency, we shall obtain, after appropriate differentiation, the expression of Planck's formula for the intensity of black-body radiation into the transparent medium, expressed in terms of the wavelength in a medium, as:

$$I_\lambda(T,\lambda) = \frac{2hc_0^2}{n^2\lambda^5}\frac{1}{[\exp(hc_0/n\lambda kT) - 1]} \tag{6.5}$$

In the SI system the intensity, presented in such a form, is measured in $\mathrm{W/(m^3\,sr)}$. Often, in the IR band especially, the wavelength is measured in micrometres; then the dimension of radiation intensity will be W/(m$^2$ sr μm). However, it is convenient to use the frequency presentation of Planck's formula (6.2) in the cases where the radiation propagates from one medium into another, since in this case the frequency remains constant and the wavelength changes.

In many practical applications (remote sensing, heat transfer, radio-astronomy) of interest is the surface density (per unit of the surface) of a spectral flux of black-body radiation determined in the form of equation (5.5) and (5.9). Substituting the spectral density value from (6.5), we have

$$q_\lambda(T) = \frac{C_1}{n^2\lambda^5} \frac{1}{[\exp(C_2/n\lambda T) - 1]}, \tag{6.6}$$

where the quantities

$$C_1 = 2\pi h c_0^2; C_2 = \frac{hc_0}{k} \tag{6.7}$$

were called the first and second radiation constants (see Table A.4).

Note that $q_\lambda(T)$ represents the amount of radiation energy emitted by the unit area of the black-body surface at temperature $T$ per unit time, in the wavelength band unit, in all directions within the limits of the hemispherical solid angle. In the SI system this quantity is measured in W/m$^3$, and if the wavelength is measured in micrometres then this quantity is measured in W/(m$^2$ μm).

Figure 6.2 presents the spectral distribution of the surface density of a monochromatic black-body radiation flux $q_\lambda(T)$, calculated by formula (6.6) for $n = 1$. In order to understand better the implication of this equation, Figure 6.2 gives the

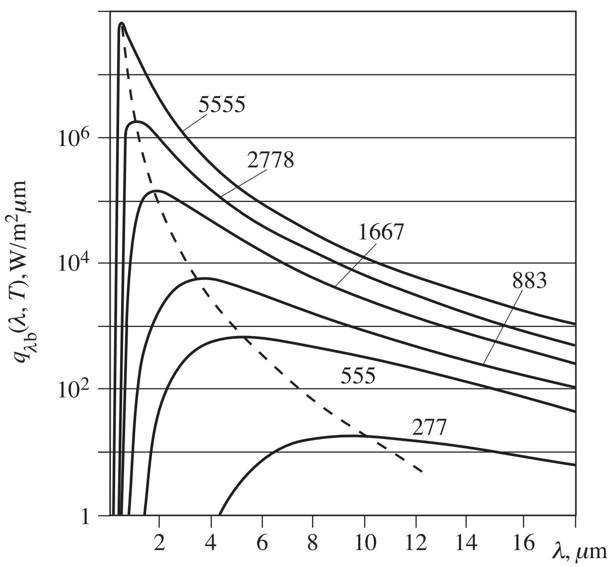


**Figure 6.2.** Hemispherical spectral radiation flux of black bodies for some values of temperatures versus wavelengths. Black-body temperatures are shown by figures next to the curves. Positions of spectral radiation flux maxima are shown by a dotted line.

wavelength dependencies of hemispherical spectral surface density of radiation flux for several values of absolute temperature. A peculiarity of Planck's curves is the increase of the energy of radiation, corresponding to all wavelengths, with increasing temperature. As was shown in section 6.1, qualitative thermodynamic considerations and everyday experience indicate that the energy of total radiation (including all wavelengths) should increase with temperature. It also follows from Figure 6.2 that this conclusion is also valid for the energy of radiation corresponding to each wavelength. Another peculiarity is the displacement of maxima of the spectral surface density of radiation flux to the side of shorter wavelengths with increasing temperature. The cross-sections of the plot in Figure 6.2 at fixed wavelengths, which determine the radiation energy as a function of temperature, allow us to state that the energy of radiation, emitted at the short-wave extremity of the spectrum, increases with temperature faster than the energy of radiation corresponding to greater wavelengths. Figure 6.2 indicates the position of the wavelength band in the visible spectrum region. For a body at temperature of 555 K only a very small fraction of energy falls on the visible spectrum range, which is virtually imperceptible by the human eye. Since the curves at lower temperatures are dropping from the red section toward the violet extremity of the spectrum, then, at first, the red light becomes visible with increasing temperature (the so-called Driper point, corresponding to 525°C). At sufficiently high temperature the emitted light becomes white and consists of a set of all wavelengths of the visible spectrum. The radiation spectrum of the Sun is similar to the radiation spectrum of a black body at a temperature of 5800 K, and a considerable portion of released energy falls on the visible spectrum range. (This type of radiation is sometimes called 'white' noise – as we see, quite wrongly.) More likely, owing to very long biological evolution, the human eye became most sensitive precisely in the spectrum region with maximum energy.

Equation (6.6) can be presented in a more convenient form that allows us to avoid constructing the curves for each value of temperature; for this purpose equation (6.6) is divided by temperature to the fifth power:

$$(q_\lambda(T,\lambda)/T^5) = \frac{\pi I_\nu(T,\lambda)}{T^5} = \frac{C_1}{(\lambda T)^5}\,\frac{1}{[\exp(C_2/\lambda T) - 1]}. \tag{6.8}$$

This equation determines quantity $q_\lambda(T,\lambda)/T^5$ as a function of single variable $\lambda T$. The plot of such a dependence is presented in Figure 6.3; it substitutes a set of curves in Figure 6.2.

The Planck law for energy distribution in the black-body spectrum gives a maximum value of the intensity of radiation that can be emitted by any body at the given temperature and wavelength. This intensity plays a part of an optimum standard, with which the characteristics of real surfaces can be compared.

But, more simply, approximate forms of the Planck law are sometimes applied. However, it is necessary to bear in mind that they can be used only in that range, where they provide acceptable accuracy.

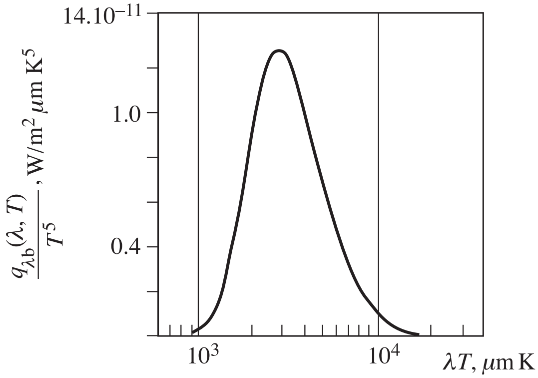


**Figure 6.3.** Hemispherical of black-body spectral radiation flux distribution versus generalized coordinates.

### 6.2.2 The Wien radiation law

If the term $\exp(C_2/\lambda T) > 1$, then equation (6.8) is reduced to the expression

$$\frac{I_\lambda(T,\lambda)}{T^5} = \frac{C_1}{\pi(\lambda T)^5 \exp(C_2/\lambda T)}, \tag{6.9}$$

which is known as the Wien radiation law. For the values of $\lambda T < 3000\,\mu\text{m K}$ this formula gives an error within the limits of 1%.

### 6.2.3 The Rayleigh–Jeans radiation law

Another approximate expression can be obtained by expanding the denominator in equation (6.8) into the Taylor series. If $\lambda T$ is essentially greater than $C_2$, then the series can be restricted by the second term of expansion, and equation (6.8) takes the form:

$$\frac{I_\lambda(T,\lambda)}{T^5} = \frac{C_1}{\pi C_2}\frac{1}{(\lambda T)^4}. \tag{6.10}$$

This equation is known as the Rayleigh–Jeans radiation law. This formula gives an error within the limits of 1% for the values of $\lambda T > 7.8 \times 10^5\,\mu\text{m K}$. These values are outside the range usually considered in IR thermal radiation, but they are of principal importance for the radio-frequency band. The frequency presentation of the Planck formula is usually applied in this band, and then the Rayleigh–Jeans law takes the widely used form:

$$I_\nu(T,\nu) = \frac{2\nu^2}{c_0^2}nkT = \frac{2f^2}{c_0^2}nkT. \tag{6.11}$$

### 6.2.4 The Wien displacement law

Another quantity of interest, which relates to the black-body radiation spectrum, is the wavelength $\lambda_m$, to which corresponds the maximum of surface density of an emitted energy flux. As is shown by the dotted curve in Figure 6.2, this maximum displaces to the side of shorter wavelengths as the temperature increases. Quantity $\lambda_m$ can be found by differentiating the Planck function from equation (2.12) and by equating the obtained expression to zero. As a result, the transcendental equation is obtained

$$\lambda_m T = \frac{C_2}{5} \frac{1}{1 - \exp(-C_2/\lambda_m T)}, \tag{6.12}$$

whose solution is as follows:

$$\lambda_m T = C_3 \tag{6.13}$$

and represents one of expressions of the Wien displacement law. The values of constant $C_3$ are given in Table A.4. According to equation (6.13), as the temperature increases, the maxima of surface density of the radiation flux and its intensity displace to the side of shorter wavelengths in inverse proportion to $T$. If we consider the black-body radiation into a transparent medium (with refractive index $n$), then the Wien law takes the form of

$$n\lambda_{m,n} T = C_3, \tag{6.14}$$

where $\lambda_{m,n}$ is the wavelength corresponding to the maximum of radiation in the transparent medium.

Of interest is the fact that the substitution of the wavelength from the Wien displacement law (6.13) into equation (6.8) results in the following expression:

$$I_\lambda(T, \lambda_m) = T^5 \frac{C_1}{\pi C_3^5[\exp(C_2/C_3) - 1]}. \tag{6.15}$$

It follows from this relation, that the maximum value of radiation intensity increases in proportion to temperature to the fifth power. Generally speaking, it is this relation that was obtained by Wien in 1893.

It can easily be seen from the expression obtained that the maximum of spectral intensity of the microwave background of the universe at radiation temperature of 2.73 K will be approximately equal to 1 mm.

### 6.2.5 The Stefan–Boltzmann law

Integrating $q_\lambda(T)$ over all wavelengths from zero to infinity (or, accordingly, $q_\nu(T)$ in the frequency presentation), we obtain by means of expressions for determinate integrals (Gradshteyn and Ryzhik, 2000) the surface density of the total black-body radiation flux $q(T)$ as:

$$q(T) = \int_0^\infty q_\lambda(T, \lambda)\, d\lambda = \int_0^\infty q_\nu(T, \nu)\, d\nu = \pi \int_0^\infty I_\nu(T, \nu)\, d\nu = n^2 \sigma T^4, \tag{6.16}$$

where the Stefan–Boltzmann constant $\sigma$ is equal (see Table A.4) to:

$$\sigma = \frac{2\pi^5 k^4}{15 c_0^2 h^3}. \tag{6.17}$$

Similar expressions can also be obtained for the total radiation intensity:

$$I(T) = \int_0^\infty I_\nu(T,\nu)\,\mathrm{d}\nu = n^2 \frac{\sigma}{\pi} T^4 \tag{6.18}$$

and for the total volume density of radiation (for vacuum):

$$u = \int_0^\infty u_\nu(T,\nu)\,\mathrm{d}\nu = aT^4, \tag{6.19}$$

where $a$ is called the radiation density constant (see Table A.4).

Let us consider now the instructive example, associated with the relation of the amount of energy, emitted from the unit of black body's surface into vacuum within the whole frequency band and in the radio-frequency band separately. Using relations (6.16), we obtain the total power, emitted by a black body from 1 square metre at room temperature (300 K), which is equal to 450 W. Now, using the Rayleigh–Jeans law (6.11), we obtain the expression for the Stefan–Boltzmann law in the long-wavelength approximation, as follows:

$$q(T) = \frac{2}{3}\frac{\pi k}{c_0^2} T\nu^3. \tag{6.20}$$

From this expression we can easily obtain the estimate for the total power emitted by a black body from 1 square metre at $T = 300$ K throughout the radio-frequency band from zero frequency up to $10^{11}$ Hz (the wavelength is 3 mm). It is equal to $10^{-4}$ W. Thus, the amount of energy falling on the whole radio-frequency band is $10^{-7}$ times lower than the total power of black-body radiation. In this case an even smaller part ($10^{-9}$) of the total power will fall on the whole, for example, centimetre band. And, in spite of such small values of radiation power in the radio-frequency band, modern microwave remote radio systems successfully record such low levels of a thermal signal (see Chapter 3).

### 6.2.6 Correlation properties of black-body radiation

From the viewpoint of the theory of random processes (Chapter 2), the spectral volume density of radiation energy $u_\nu(\nu)$ represents a spectral density of fluctuating strengths $E(t)$ and $H(t)$ of the thermal radiation field. This can easily be seen, taking into consideration relations (1.17), (2.27), (5.13). In each of the planar waves into which this field can be decomposed, the relation between the vectors of running planar waves is given by expression (1.11), all directions of strengths being equiprobable. As a result of small transformations in (1.17), it can be seen that the electric and magnetic energies are equal, and $E$ and $H$ components in any arbitrary direction have identical correlation functions but are not correlated among themselves. Thus, the correlation properties can be considered with respect to any component of the electromagnetic field strength.

Let us find the correlation coefficient corresponding to the spectral density (6.1), i.e. the quantity

$$R_u(\tau) = \frac{B_u(\tau)}{B_u(0)}, \tag{6.21}$$

where

$$B_u(\tau) = \frac{2}{\pi}\int_0^\infty u_\nu(T,\nu)\cos 2\pi\nu\,\mathrm{d}\nu \tag{6.22}$$

Substituting here expression (6.1) for the spectral density and calculating the integral, we obtain (Rytov, 1966):

$$R_u(\tau) = 15\left[\frac{3}{\mathrm{sh}^4\beta} - \frac{3}{\beta^3} + \frac{2}{\mathrm{sh}^2\beta}\right] = \frac{15}{2}\frac{\mathrm{d}^3}{\mathrm{d}\beta^3}L(\beta), \tag{6.23}$$

where $L(\beta) = \mathrm{cth}\beta - (1/\beta)$ is the Langevin function, and $\beta = 2\pi^2 kT\tau/h$ (sh$x$, cth$x$ are hyperbolic sine and cotangent).

The form of the correlation coefficient from the temporary lag is shown in Figure 6.4 and, as should be expected, it certainly does not look like the delta-function. First of all, we note that for $\beta \cong 1.37$ (which corresponds to $\tau_0 = h/2\pi kT$) the positive correlation is changed to a negative one. This implies that for temporary shifts $\tau < \tau_0$ the values of component $E_p(t)$ in some fixed direction $p$ will more frequently have at instants $t$ and $t + \tau$ the same sign, and for $\tau > \tau_0$ the opposite sign. The temporary lag $\tau_0$ can be put in correspondence to the spatial correlation radius $\lambda_0 = c\tau_0$, which to an accuracy of numerical coefficient coincides with the wavelength $\lambda_m = 0.2(hc/kT)$ in the Wien displacement law. From the comparison of these expressions we can obtain the following important relation:

$$\lambda_0 = 0.35\lambda_m. \tag{6.24}$$

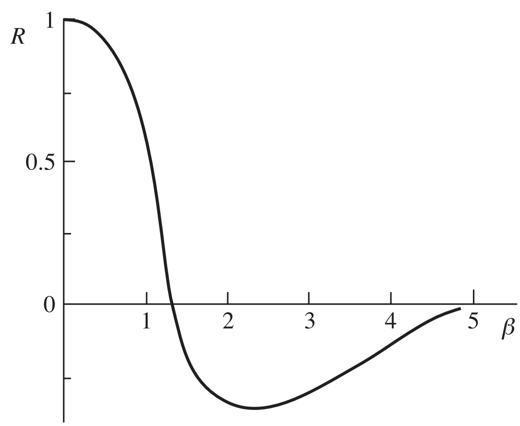


**Figure 6.4.** Correlation coefficient of spectral black-body radiant energy density versus generalized lag $\beta = 2\pi^2\dfrac{kT\tau}{h}$.

It can easily be found from this relation that the spatial correlation radius of the microwave background of the universe equals a quite macroscopic value, namely, $\lambda_0 = 0.35$ mm. It is interesting to mention, that earlier (in 1971) it was proposed to measure the velocity of solar system motion relative to the microwave ('absolute ether at rest') background by recording the variable part of the interferogram (in other words, the correlation coefficient (6.24)) just at the place where it changes its sign (near $\lambda_0$) (Soglasnova and Sholomitskii, 1971).

At spatial distances greater than (4–5)$\lambda_0$ the correlation sharply drops, and the statistical process of emission at such scales can be represented as a non-correlated random (white) noise. Generally speaking, it is this circumstance which is often used in analysing thermal radiation.

## 6.3 THE KIRCHHOFF LAW

As we have noted above (Chapter 4), the fluctuation–dissipation theorem, which represents one of the fundamental laws of statistical physics, establishes for an arbitrary dissipative physical system the relationship between the spectral density of spontaneous equilibrium fluctuations and its nonequilibrium properties and, in particular, the energy dissipation in a system. For the wave field of an absorbing half-space, i.e. for the field of radiation which can be recorded by an external (relative to the emitting medium) instrument, the solution of the fluctuation electrodynamic problem directly results in the Kirchhoff law in the form of (4.20).

Before describing the properties of non-black physical bodies, it is useful to introduce the definitions of emissive ability and absorbing ability and also to consider the Kirchhoff law forms that are often used for analysing the emitting half-space (i.e. when there are two material media with a sharp boundary between them), as well as for analysing the radiation transfer processes in a transparent infinite medium (the atmosphere). In the first case (the planar version) the measuring instrument is inside one of media and measures the radiation of the other one. In the second case (the solid version) the instrument is directly inside the medium, whose radiation it just measures. Below we shall consider the first version in detail. As far as the solid (three-dimensional) version is concerned, we shall postpone the detailed study of radiation transfer processes for this version until Chapter 9.

### 6.3.1 Emissive ability

This characteristic, which is sometimes called the emissivity, indicates what portion of black-body radiation energy constitutes the radiation energy of a given body. The emissive ability of a real physical body depends on such factors as its temperature, its physical and chemical composition, its intrinsic geometrical structure, its degree of surface roughness, the wavelength to which the emitted radiation corresponds, and the angle at which the radiation is emitted. For remote microwave sensing problems it is necessary to know the emissive ability both in any required direction (the angular characteristics) and at various wavelengths (the spectral characteristics). In this case the degree of remote information capacity of angular and spectral characteristics is strongly distinguished, generally speaking, depending on the type of a physical object under study. This radiation characteristic is called the directional emissive ability (or the directional emissivity).

In calculating a body's total energy losses through radiation (as in heat-and-power engineering problems) it is necessary to know the radiation energy in all directions and, for this reason, the emissivity, averaged over all directions and wavelengths, is used in such calculations. For calculating a complicated heat exchange through radiation between surfaces, the emissivities can be required, which are averaged only over the wavelengths and not over the directions. So, the researcher should possess the emissivity values, averaged in different ways, and they should be obtained, most frequently, from the available experimental data.

In this book we shall keep to the definition of directional spectral emissivity. If necessary, this emissivity can then be averaged over the wavelengths and the directions, and, finally, over wavelengths and directions simultaneously. Averaged over the wavelengths, they are called total (integral) quantities, and the quantities averaged over the directions are called hemispherical quantities (Siegel and Howell, 1972).

Recalling the definitions of spectral intensity of emission from the unit of a physical body's surface (see section 5.1), we shall define the directional emissivity as the ratio of the spectral intensity of a real surface $I_\nu(\mathbf{r}, \Omega, T, \ldots)$, which depends on body's temperature, physical and chemical composition, intrinsic geometrical structure and degree of surface roughness, as well as on the observation angle and working wavelength (frequency), to the black-body intensity $I_{\nu B}(\nu, T)$ at the same temperature and at the same wavelength (frequency) (6.2):

$$\kappa_\nu(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots) = \frac{I_\nu(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots)}{I_{\nu B}(T, \nu)} \tag{6.25}$$

This expression for emissivity is most general, since it includes the dependencies on the wavelength, direction and temperature. The total and hemispherical characteristics can be obtained by appropriate integration (Siegel and Howell, 1972).

As far as the volumetric version is concerned, here we should note that the directional spectral emissivity of a unit homogeneous volume of the medium is equal to the ratio of intensity of radiation, emitted by this volume in the given direction, to the intensity of radiation emitted by a black body at the same temperature and wavelength.

### 6.3.2 Absorbing ability

The absorbing ability of a body is the ratio of the radiation flux absorbed by the body to the radiation flux falling (incident) on the body. The incident radiation possesses the properties inherent in a particular power source. The spectral distribution of the incident radiation energy does not depend on temperature or on the physical nature of an absorbing surface (so long as the radiation, emitted by the surface, is not partially reflected back onto this surface). In this connection, in defining the absorbing ability (as compared to emissivity), additional difficulties arise, which are related to the necessity of taking into account the directional and spectral characteristics of incident radiation.

By the directional spectral absorptivity $\alpha(\mathbf{r}, \mathbf{\Omega}, T, \ldots)$ we shall mean the ratio of the spectral intensity of absorbed radiation $I_{\nu a}(\mathbf{r}, \mathbf{\Omega}, \nu, T, \ldots)$ to the spectral intensity of incident radiation at the given wavelength and from the given direction $I_{\nu 0}(\mathbf{r}, \mathbf{\Omega}, \nu, T, \ldots)$:

$$\alpha_\nu(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots) = \frac{I_{\nu a}(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots)}{I_{\nu 0}(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots)}. \tag{6.26}$$

In addition to the incident radiation dependence on the wavelength and direction, the directional spectral absorptivity is also a function of temperature, physical and physico-chemical properties of an absorbing surface.

### 6.3.3 The Kirchhoff law forms

This law establishes the relation between the abilities of emitting and absorbing the electromagnetic energy by any physical body. This law can be presented, to an equal degree of certainty, in terms of spectral, integral, directional or hemispherical quantities. In the case of microwave sensing it is expedient for us to dwell on the directional properties. From equations (5.1) and (6.25), the energy of radiation, emitted by a surface element from $\mathrm{d}A$ in the frequency band $\mathrm{d}\nu$, within the limits of solid angle $\mathrm{d}\Omega$ and during time $\mathrm{d}t$, is equal to

$$\mathrm{d}E_\nu = \kappa_\nu(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots) I_{\nu B}(T, \nu)\, \mathrm{d}A \cos\theta\, \mathrm{d}\Omega\, \mathrm{d}\nu\, \mathrm{d}t. \tag{6.27}$$

If we assume the element $\mathrm{d}A$ at temperature $T$ to be inside the isothermal, ideally black, closed cavity, also at temperature $T$, then the intensity of radiation, falling on the element $\mathrm{d}A$ in the direction $\mathbf{\Omega}$, will be equal to $I_{\nu B}(T, \nu)$ (remember the property of isotropy of radiation intensity of an ideal black cavity) (section 6.1). For maintaining the isotropy of radiation inside an ideal black closed cavity, the fluxes of absorbed and emitted radiation, determined by equations (6.26) and (6.27), should be equal and, therefore, the following relation should be met:

$$\kappa_\nu(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots) = \alpha_\nu(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots). \tag{6.28}$$

Equality (6.28) sets the relationship between the fundamental properties of physical substances and is valid, without limitations, for all media in a state of thermodynamic equilibrium. It represents the most general form of the Kirchhoff law. It is just this form of the law that was presented by G. Kirchhoff in his famous work published in 1860 (see Schopf (1978) for more details).

The following important corollary follows from (6.28). Since in its physical sense quantity $\alpha$ is always less than unity, the emissive ability of any physical body is concluded between zero and unity, i.e. $0 < \kappa < 1$. This characteristic is used very widely in microwave sensing, since it allows us to estimate and compare the emission properties of investigated substances without resorting to the measurement of radiation energy values.

Another formulation of Kirchhoff's law, also set forth by him, is also possible. The ratio of the radiation intensity of a physical body, heated up to temperature $T$, to its absorptivity is a universal function of temperature and frequency, which does not depend on the physical and geometrical properties of a body. Proceeding from (6.25), (6.26) and (6.28), we have:

$$\frac{I_\nu(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots)}{\alpha_\nu(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots)} = I_{\nu B}(T, \nu). \tag{6.29}$$

Kirchhoff himself considered the finding of an explicit form of this universal function to be 'the problem of fundamental importance' for physics (Schopf, 1978).

One further form of Kirchhoff's law is used in microwave sensing (and we shall use it later). It follows from the relations presented above:

$$I_\nu(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots) = \kappa(\mathbf{r}, T, \nu, \mathbf{\Omega}, \ldots) I_{\nu B}(T, \nu). \tag{6.30}$$

As we have seen, in section 1.4, electromagnetic waves propagate in free space, where there exist two components of a wave, which oscillate at right angles with respect to each other and with respect to the wave propagation direction. In the particular case of equilibrium thermal radiation these two components of polarization are equal. Strictly speaking, relations (6.28)–(6.31) are fulfilled for each polarization component, and, in order that they be valid for the total incident radiation, the radiation should have equal polarization components. Thus, the original equilibrium radiation is non-polarized (which, however, is invalid for grey bodies) (see Chapter 7).

The Kirchhoff law was proved for the case of thermodynamic equilibrium in an isothermal closed cavity and, hence, it is strictly valid only in the absence of a resulting thermal flux directed towards the surface or away from it. Under real conditions, as a rule, there exists a resulting flux of electromagnetic radiation, so that relations (6.28) and (6.30) are approximate, strictly speaking. The validity of this approximation is confirmed by reliable experimental data, according to which in the majority of practical cases the ambient radiation field does not have any significant influence on the values of emissive and absorbing abilities. Another confirmation of this approximation is a substance's ability to be at the state of local thermodynamic equilibrium (section 4.4), in which the set of energy states during absorption and emission processes corresponds, to a very close approximation, to their equilibrium distributions (corresponding to the local temperature in this case). Thus, the spreading of Kirchhoff's law to natural nonequilibrium systems is not the result of simple thermodynamic considerations, but, most likely, it is a result of the physical nature of substances. Owing to this circumstance, in the majority of cases the substance is capable of independently maintaining a local thermodynamic equilibrium and, thus, to possesses 'independence' of its properties from the ambient radiation field.

In conclusion, we note that, as astrophysical investigations have shown, the Kirchhoff law can actually also be applied in cases where the radiation is not in full equilibrium with the substance, and its distribution over frequencies essentially differs from Planck's one. However, the Kirchhoff law is not applicable in cases where thermodynamic equilibrium conditions are strongly violated (nuclear explosions, shock waves, the interplanetary medium). This law is not suitable for determining the emissivities of sources of non-thermal radiation (synchrotron, maser radiation, thunderstorm activity) and sources of quasi-deterministic radiation (radio- and TV-broadcasting, communications).