9

Foundations of radiative transfer theory

This chapter presents the phenomenological basis and principal energy considerations underlying radiative transfer theory (RTT). The analysis of basic equations and fundamental concepts, required for studying radiative transfer in absorbing, emitting and scattering media, are given in this chapter. The formal and approximate solutions of the equation of radiative transfer, given in this chapter, are widely used in subsequent chapters when considering radiative transfer in dispersed media (hydrometeors and aerosols in the atmosphere). Attention is chiefly given to the analysis of solutions of the transfer theory intended for studying thermal radiation processes in the microwave band.

9.1 RADIATIVE TRANSFER THEORY PHENOMENOLOGY

Although the principal solution of thermal radiation problems is possible using the fluctuation–dissipation theorem (see Chapter 4), the practical solution of many problems is rather complicated and requires the application of other physical approaches. First of all, one should mention here the energy approach associated with particular phenomenological concepts developed in studying electromagnetic radiative transfer in absorbing, emitting and scattering media. The phenomena of energy transfer by radiation in media, which can absorb, emit and scatter radiation, have been of interest for a long time. This interest was aroused by the study of complicated and interesting phenomena related to astrophysical problems, remote sensing, nuclear explosions, flows in hypersonic compressed layers, rocket engines and plasma generators designed for nuclear fusion. Although some of these applications appeared quite recently, the absorption and emission processes in gases have aroused interest for more than a hundred years. One of the first investigations was devoted to electromagnetic radiation absorption by the terrestrial atmosphere. This problem has always stirred the optical astronomers, who have observed the light from the Sun and more distant stars. The distorted spectra of black-body radiation of the Sun in the near-IR band, obtained for a number of years beginning with 1880 (see Siegel and Howell, (1972) for more details), have testified to essential wavelength dependence of emissive properties of gases in the terrestrial atmosphere. Solar radiation absorption by the cloudless atmosphere was caused, as was found later, primarily by water vapour and carbon dioxide existing in the atmosphere. As a matter of fact, these investigations were the first systematic remote studies of the Earth's atmosphere using the 'regular transmission' technique.

The emission of gases (and, later, of plasma systems) has also been of interest for astrophysicists in connection with studies of the structure of stars. Models of stellar atmospheres and of the Sun were proposed describing the energy transfer processes in these objects, after which the emission spectra, calculated on the basis of these models, have been compared with those obtained experimentally. It was these investigations, on the basis of which the phenomenological foundations were developed and radiative transfer theory was constructed and advanced both for astronomy and for remote sensing applications (Troitskii, 1954; Chandrasekhar, 1960; Sobolev, 1963; Staelin, 1969; Kondratyev and Timofeev, 1970; Malkevich, 1973; Marchuk, 1976; Marchuk *et al.*, 1986; Sabins, 1987; Apresyan and Kravtsov, 1996; Thomas and Stamnes, 1999; Matzler, 2000). An overwhelming majority of the physical results obtained in astronomy, radio-astronomy and remote sensing was based, one way or another, on using the methodology of radiative transfer theory.

In industry the problem of emission of gases became topical in the 1920s in connection with studying heat exchange in furnaces (smelting of steel and glass), in combustion chambers of engines and later, in the 1950s and 1960s, in rocket engines (Ozisik, 1973; Siegel and Howell, 1972). At the same time, it was found that similar physical approaches and, accordingly, the equations can govern the processes of the propagation of neutrons in nuclear reactors (Murray, 1957). This undoubtedly gave a new impetus to the detailed investigation of transfer processes.

In studying radiative transfer in absorbing, emitting and scattering media, two very important features arise. First, in such media the absorption and emission of radiation occur not only at the boundaries of a system, but also at every point inside a medium. The same is true for scattering. For complete solution of the energy transfer problem it is necessary to know the volume field of temperature and physical properties of a medium at each point of a system. By a point of a system here is meant a physically infinitesimal (unit) volume of a medium, which contains a fairly large number of particles, the interaction between which can provide the local thermal equilibrium conditions (see Chapter 4). By particles here is meant either a set of macroscopic particles (aerosols, water drops, snow and ice particles, volcanic ashes or particles of another nature), or a set of quantum particles (atoms and molecules of gases).

In the first version transfer theory is considered at the macroscopic level bringing in the results of Maxwell's theory of scattering on macroscopic particles (Mie scattering theory). The properties of a physical medium, in which the process takes place, are taken into account by a set of some (phenomenological in a certain sense) coefficients determined either experimentally or by calculations.

In the second version the electromagnetic field is considered as a combination of particles (the 'photon gas'), and the radiative transfer process of interest is considered as the interaction of these particles with the particles of substance on the basis of quantum-mechanical concepts (see Chapter 11 for more details).

To find the local values of radiation intensity in a medium the 'astrophysical' approach is applied (Siegel and Howell, 1972), where the complete equation of radiative transfer is solved. As it will be shown later, the radiation intensity is related to energy transferred along some selected direction. Having determined the variation of the intensity of radiation along the path of its propagation, we can get an idea of how absorption, emission and scattering processes influence radiative transfer. Such an approach is most efficient in considering the problems related to absorption and emission of the terrestrial atmosphere and to the structure of stars, and in other problems where the quantity sought is the spectral intensity of radiation at a point of the medium. The solution of a complete problem, as we shall see later, encounters considerable mathematical difficulties.

Second, the spectral characteristics of rarefied systems (gases) have much sharper changes (usually narrow lines of various types), than the spectral characteristics of solid or liquid bodies. Such a kind of noise emission is called selective radiation (see Chapter 11). The physical nature of this circumstance is well known now: it lies in the features and distinctions of the quantum-mechanical structure of gases and solid bodies. Therefore, for studying the radiation of gaseous media it is necessary, as a rule, to consider in detail the spectral characteristics (i.e. the so-called radio spectroscopy methods should be applied). In using approximations, based on the properties averaged over the spectrum, special caution is required. The majority of simplifications, which are introduced in solving the radiation problems of physical media, are made with the purpose of avoiding these difficulties. So the 'astrophysical' approach often undergoes serious simplification to facilitate its use in engineering calculations carried out mainly with the purpose of determining the integral (over frequencies or solid angles) of energy fluxes, rather than the differential radiation intensity (Ozisik, 1973; Siegel and Howell, 1972). However, in solving remote sensing problems such simplifications are inadmissible, since in the majority of cases very important information on polarization and the spectral characteristics of investigated objects is lost.

9.1.1 RTT applicability conditions

The use of RTT in relation to real media is based on some physical simplifications which allow us, generally speaking, to advance in studying radiative transfer in composite (for example, multiphase) media, where the direct use of the Maxwell theory is troublesome. It should be mentioned here that, in the majority of works on the presentation of the RTT fundamentals and application of the theory, the physical suppositions underlying this theoretical presentation are as a rule neither discussed nor analysed.

(1) The use of the geometric optics approximation, where the electromagnetic radiation wavelength is essentially lower than the scale of variation of a macrosystem's parameters. This approximation, as known, uses the beam conceptions for electromagnetic wave propagation in a medium (see section 1.6).
(2) The use of the approximation of electromagnetic rarefaction of a medium, where the distance between particles, constituting an elementary volume of a medium, essentially exceeds the working wavelength. The original flux, falling on an elementary volume, reaches each particle. The particles do not 'shade' each other, and there is no mutual interference between the particles. Thus, the total effect of electromagnetic field interaction with a group of particles is achieved by summing the interaction effects on each particle.
(3) The relationship between the size of individual particles and the working wavelength is arbitrary, i.e. all diffraction effects at the electromagnetic field interaction with an individual particle should be taken into account.
(4) All processes of the external electromagnetic field interaction with a unit volume of a medium are reduced to three acts only – the absorption act, the emission act and the scattering act (see section 9.2 for more details).

These conditions will be referred to and analysed at the appropriate places in the presentation of the foundations of RTT.

9.2 ENERGY TRANSFORMATION BY A VOLUME ELEMENT

Consider a beam of radiation with intensity $I_\nu(\mathbf{r}, \mathbf{\Omega})$ propagating in the absorbing, emitting and scattering medium in a given direction. The energy of radiation will decrease owing to its absorption by substance and owing to the deviation of a part of the radiation from the initial trajectory as a result of scattering in all directions. But, at the same time, the energy will increase because of thermal radiation emission by the substance volume. The absorption, scattering and emission of radiation by a substance have effect on the energy of a radiation beam that propagates in it. In this case the total balance of change of the initial intensity can be, certainly, both positive and negative. Besides, a strong inhomogeneity of the energy balance, both over the substance volume and over the observation direction, is possible. These properties have been analysed in detail in books by Chandrasekhar (1960), Sobolev (1963), Ozisik (1973), Siegel and Howell (1972). In this section we briefly consider radiation interaction with a volume element using the phenomenological concepts of three acts of radiation interaction with the substance volume element – the act of absorption, the act of emission and the act of scattering.

9.2.1 The act of absorption

Consider a beam of monochromatic radiation with intensity $I_\nu(\mathbf{r}, \mathbf{\Omega}')$ restricted by the elementary solid angle $\mathrm{d}\Omega'$ and falling along the normal to the element of surface $\mathrm{d}A$ of the layer of width $\mathrm{d}S$ (Figure 9.1). As the incident radiation propagates in the

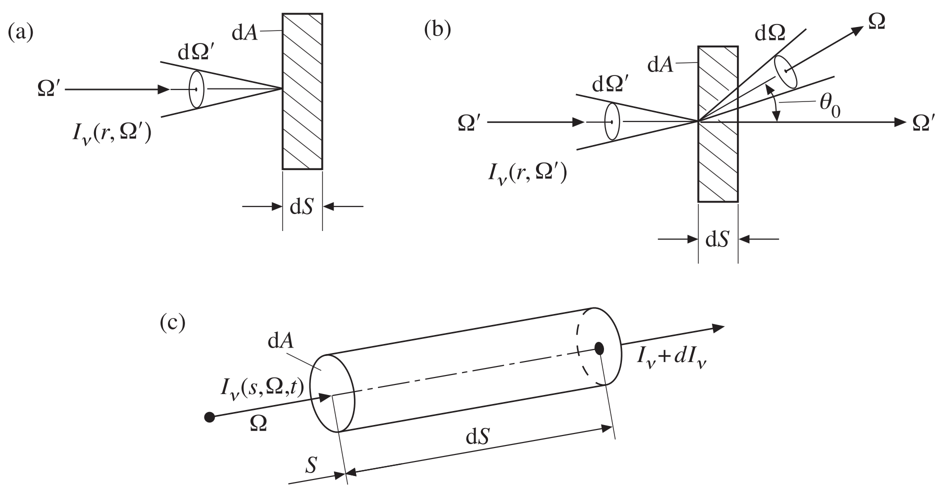

Figure 9.1. Schematic presentation of the geometry of the radiative transfer procedure by a volume element: (a) the act of absorption; (b) the act of scattering; (c) the total energy transformation.

substance, the part of radiation is absorbed by this substance. Designate by $\gamma_\nu(r)$ the spectral absorption coefficient, which is equal to the fraction of incident radiation, absorbed by substance over the unit of the radiation propagation path length, and has the dimension of (length)$^{-1}$. Then the quantity

$$\gamma_\nu(\mathbf{r}) I_\nu\big(\mathbf{r}, \mathbf{\Omega}'\big)\,\mathrm{d}\Omega' \tag{9.1}$$

characterizes the absorption of the incident radiation $I_\nu(\mathbf{r}, \mathbf{\Omega}')$ by substance from the direction Ω' per unit of time, in a unit of volume $\mathrm{d}A\,\mathrm{d}S$ and in a unit frequency band.

If the radiation falls on a volume element from all directions within the limits of a total solid angle, then expression (9.1) should be integrated over all solid angles (see section 5.1):

$$\gamma_\nu(\mathbf{r}) \int_0^{2\pi} \int_{\mu'=-1}^{+1} I_\nu(\mathbf{r}, \mu', \varphi')\,\mathrm{d}\mu'\,\mathrm{d}\varphi'. \tag{9.2}$$

This expression characterizes the absorption, by substance, of radiation falling on a separated volume element from all directions within the limits of a spherical space, per unit of time, in a unit of volume and in a unit frequency band (with the dimension of $\mathrm{W/m^3\,Hz}$).

9.2.2 The act of emission

In the problems of radiative transfer in absorbing, emitting and scattering media the supposition of a local thermal equilibrium (LTE) is used almost always with the purpose of deriving the expression for the intensity of thermal radiation of a volume element (see section 4.4). In essence, the question here is about the volumetric form of Kirchhoff's law (see section 6.3). The LTE conditions imply that any small volume element of a medium is at the local thermodynamic equilibrium, owing to which the state of any point can be characterized by local temperature $T(\mathbf{r})$. This supposition is lawful in the case where the collisions of particles in a substance occur so frequently that they result in a local thermodynamic equilibrium at each point $\mathbf{r}$ of a medium. In this case the emission of radiation by a volume element can be described by means of the volumetric form of Kirchhoff's law. If we designate by $J_\nu(\mathbf{r})$ the radiation emitted by a unit volume of substance per unit of time, within the limits of a unit solid angle and in a unit frequency band (with dimension of $\mathrm{W/(m^3\,Hz\,St)}$), then the emission of radiation by substance can be expressed in terms of the Planck function for the intensity of radiation of an ideal black body:

$$J_\nu(\mathbf{r}) = \gamma_\nu(\mathbf{r}) I_{\nu\mathrm{B}}[\nu, T(\mathbf{r})], \tag{9.3}$$

where $I_{\nu\mathrm{B}}(T)$ is determined by formula (6.2).

If the supposition of a local thermodynamic equilibrium is inapplicable for the studied system (this requires some special investigation), then the emission of radiation by a substance becomes a function of energetic states in the system, and the problem of radiative transfer in such media is essentially complicated.

9.2.3 The act of scattering

If the medium includes inhomogeneities in the form of small particles, then the radiation beam, while passing through this medium, will be scattered in all directions. For example, particles of dust or drops of water in the atmosphere scatter electromagnetic waves passing through such a medium, as well as the thermal radiation formed in other spatial parts of a medium. Thus, the general picture of thermal radiation of the whole medium may be rather complicated.

Absolutely homogeneous media do not exist in nature, except in an absolute vacuum. However, in many cases the medium can be considered to be optically (or electromagnetically) homogeneous, if the linear size of inhomogeneities is known to be considerably smaller than the radiation wavelength. For example, a cloudless atmosphere in the microwave band satisfies these conditions. One should also distinguish coherent from incoherent scattering. The scattering is called coherent if the scattered radiation has the same frequency as the incident radiation, and incoherent if the frequency of scattered radiation differs from that of the incident radiation – owing to turbulent motion of macroscopic particles in air, for instance. As a whole, the scattering problem is very complicated. A considerable literature is devoted to studying these problems (see, for instance, Ishimaru, (1978, 1991)). In this chapter we shall consider only the simplest version of coherent single scattering. Nevertheless, the scattering of microwave radiation in numerous real media is well described within the framework of this approximation.

Consider a beam of monochromatic radiation with intensity $I_\nu(\mathbf{r}, \mathbf{\Omega}')$, which propagates in the direction $\mathbf{\Omega}'$ within the limits of elementary solid angle $\mathrm{d}\Omega'$, whose axis coincides with the chosen direction, and falling along the normal to the surface $\mathrm{d}A$ of the elementary layer $\mathrm{d}S$ (Figure 9.1(b)). While the incident radiation passes through the medium, part of it is scattered by substance. Designate by $\sigma_\nu(\mathbf{r})$ the spectral scattering coefficient, which is equal to the fraction of incident radiation scattered by substance in all directions over the unit of length of the propagation path of radiation and having the dimension of (length)$^{-1}$. Then the quantity

$$\sigma_\nu(\mathbf{r}) I_\nu(\mathbf{r}, \mathbf{\Omega}')\, d\Omega' \tag{9.4}$$

characterizes the scattering, by substance, of the incident radiation $I_\nu(\mathbf{r}, \mathbf{\Omega}')\, d\Omega'$ in all directions per unit of time, in a unit of volume and in a unit frequency band. In other words, this is the part of the energy which will be completely extracted from the external radiation beam falling on a unit volume in the direction Ω'. However, expression (9.4) does not provide complete information on the distribution of scattered radiation over the directions. The distribution over the directions can be described by means of the phase function (or scattering indicatrix) $p_\nu(\mathbf{\Omega}' \to \mathbf{\Omega})$, normalized in such a manner, that

$$\frac{1}{4\pi} \iint_{\Omega=4\pi} p_\nu(\mathbf{\Omega}' \to \mathbf{\Omega})\, d\Omega = 1. \tag{9.5}$$

Note that the quantity

$$\frac{1}{4\pi} p_\nu(\mathbf{\Omega}' \to \mathbf{\Omega})\, d\Omega \tag{9.6}$$

has an important physical sense: it determines the probability of the fact, that the radiation, falling in the direction $\mathbf{\Omega}'$, will be scattered within the limits of the elementary solid angle $d\Omega$ in the direction $\mathbf{\Omega}$, i.e. in the direction of observation. Then the quantity

$$[\sigma_\nu(\mathbf{r}) I_\nu(\mathbf{r}, \mathbf{\Omega}')\, d\Omega'] \frac{1}{4\pi} p_\nu(\mathbf{\Omega}' \to \mathbf{\Omega})\, d\Omega \tag{9.7}$$

characterizes the scattering, by substance, of the incident radiation per unit of time, in a unit of volume, in a unit frequency band within the limits of elementary solid angle $d\Omega$ with axis $\mathbf{\Omega}$. In other words, we have the case, where the radiation, falling on the volume from the direction $\mathbf{\Omega}'$, is re-scattered by a studied unit volume of substance in the direction of observation $\mathbf{\Omega}$. When the radiation falls on a volume element from all directions within the limits of a spherical solid angle, the integration of (9.7) over all incident solid angles gives the expression

$$\frac{1}{4\pi} \sigma_\nu(\mathbf{r})\, d\Omega \iint_{\Omega'=4\pi} I_\nu(\mathbf{r}, \mathbf{\Omega}') p_\nu(\mathbf{\Omega}' \to \mathbf{\Omega})\, d\Omega', \tag{9.8}$$

which characterizes the scattering of radiation, which falls on a volume element from all directions within the limits of a spherical solid angle and scattered within the limits of elementary solid angle $d\Omega$ with the observation axis $\mathbf{\Omega}$ per unit of time, in a unit of volume and in a unit frequency band. From the viewpoint of an external observer, this part of the radiation has the character of an extra source of radiation, which is seen from observer's direction. As we shall see soon, it is this integral which presents basic mathematical difficulties in solving the problems of radiative transfer in scattering media.

In the case, where the scattering particles of a medium are homogeneous and isotropic, and possess spherical symmetry, and there is no preferential direction of scattering in a medium, the scattering indicatrix depends only on the angle θ_0 between the directions $\mathbf{\Omega}'$ and $\mathbf{\Omega}$. It follows from the geometrical considerations (Gradshteyn and Ryzhik, 2000), that the angle θ_0 between the incident and scattered beams is determined by the expression

$$\cos\theta_0 = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi') \tag{9.9}$$

or

$$\mu_0 = \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-(\mu')^2}\cos(\varphi-\varphi'), \tag{9.10}$$

where θ, φ and θ', φ' are polar coordinates determining the directions $\mathbf{\Omega}'$ and $\mathbf{\Omega}$, and μ, μ' and μ_0 are equal to $\cos\theta$, $\cos\theta'$ and $\cos\theta_0$, respectively.

When the scattering indicatrix depends only on the angle θ_0, expression (9.8) takes the form of

$$\frac{1}{4\pi}\sigma_\nu(\mathbf{r})\,\mathrm{d}\Omega\int_0^{2\pi}\int_{-1}^{+1} I_\nu(\mathbf{r},\mu',\varphi')p_\nu(\mu_0)\,\mathrm{d}\mu'\,\mathrm{d}\varphi', \tag{9.11}$$

where μ_0 is determined by formula (9.10).

The simplest scattering indicatrix for the case of isotropic (ideal) scattering is as follows:

$$p_\nu(\mu_0) = 1. \tag{9.12}$$

Thus, the total radiation emitted by the volume element per unit of time, recalculated for a unit of volume, in a unit frequency band and within a unit solid angle, whose axis represents the given direction $\mathbf{\Omega}$, consists of thermal radiation and scattered radiation and can be presented as

$$J_\nu(\bar{r}) + \frac{1}{4\pi}\sigma_\nu\iint_{\Omega'=4\pi} I_\nu(\mathbf{r},\mathbf{\Omega}')p_\nu(\mathbf{\Omega}'\to\mathbf{\Omega})\,\mathrm{d}\Omega'. \tag{9.13}$$

If Kirchhoff's law is valid and the medium does not have preferential direction of scattering, this expression takes the following form:

$$\gamma_\nu(\mathbf{r})I_{\nu\mathrm{B}}[T(\mathbf{r})] + \frac{1}{4\pi}\sigma_\nu(\mathbf{r})\int_0^{2\pi}\int_{-1}^{+1} I_\nu(\mathbf{r},\mu',\varphi')p_\nu(\mu_0)\,\mathrm{d}\mu'\,\mathrm{d}\varphi'. \tag{9.14}$$

Here the first term describes thermal radiation emitted by heated substance of a unit volume, and the second term describes the radiation falling on the same volume element from all directions within the limits of a spherical solid angle and scattered in the observation direction $\mathbf{\Omega}$.

Thus, as a result of the main radiation beam interaction with a unit volume of substance, two (conventionally, positive) radiation components will exist, which put the energy into the main flux recorded by an observer, and two components (conventionally, negative), which extract the energy from the main flux. As we have already noted, the total balance of change of the initial intensity can be, certainly, both positive and negative, depending on the relationship between the processes of emission, scattering and absorption in a unit volume.

The scattering concept described above is called a single scattering regime. There exist, of course, other approaches to the description of scattering process, for example, taking into account multiple scattering. The study of such approaches is a subject for separate consideration, however; so we refer the interested reader to the specialized literature (Ishimaru, 1978, 1991).

The introduction of definitions for spectral absorption and scattering coefficients and scattering indicatrix in this paragraph was done in a purely phenomenological manner. The next important stage is the problem of attributing the values of the introduced coefficients to the structure of substance (for example, formed as a cloud of water particles). This procedure will be performed in Chapter 10. Now we shall proceed to deriving the basic equation of radiative transfer theory.

9.3 THE RADIATIVE TRANSFER EQUATION

The spatial-angular distribution of radiation intensity $I_\nu(\mathbf{r}, \mathbf{\Omega})$ in a studied medium satisfies the so-called radiative transfer equation. Quite various approaches can be used for deriving this equation. It can be obtained using rigorous methods of statistical physics, by using the Boltzmann equation for radiative transfer as a transfer of photon gas. On the other hand, it is possible to use energy considerations, writing the energy balance equation for some elementary volume on the beam propagation path (Chandrasekhar, 1960; Sobolev, 1963; Ozisik, 1973). An equivalent equation was obtained in the theory of transfer of neutrons (Murray, 1957). We shall make use of the energy approach as most instinctive physically.

Consider the emitting, absorbing and scattering medium characterized by the spectral absorption coefficient $\gamma_\nu(\mathbf{r})$ and spectral scattering coefficient $\sigma_\nu(\mathbf{r})$. The beam of monochromatic radiation with intensity $I_\nu(\mathbf{r}, \mathbf{\Omega})$ propagates in this medium in the observation direction Ω along the path s. We choose the elementary volume in the form of a cylinder with cross section $\mathrm{d}A$, length $\mathrm{d}s$, disposed in the vicinity of coordinate s, the axis of a cylinder coinciding with the direction of s (Figure 9.1(c)). (As subsequent investigations have shown, the form of a unit volume does not play any part in deriving the basic equation.) Let $I_\nu(s, \mathbf{\Omega})$ be the radiation intensity at point s, and $I_\nu(s, \mathbf{\Omega}) + \mathrm{d}I_\nu(s, \mathbf{\Omega})$ be the radiation intensity at point $s + \mathrm{d}s$, and $\mathrm{d}I_\nu$ be the variation (positive or negative) of the intensity flux when it passes the path $\mathrm{d}s$.

The quantity

$$\mathrm{d}I_\nu(s, \mathbf{\Omega})\,\mathrm{d}A\,\mathrm{d}\Omega\,\mathrm{d}\nu\,\mathrm{d}t \tag{9.15}$$

represents the difference of energies of radiation, which intersects the surface $\mathrm{d}A$ at points $s + \mathrm{d}s$ and s for the time interval $\mathrm{d}t$ in the vicinity of t, in the frequency band $d\nu$ in the vicinity of ν, and propagates within the limits of a unit solid angle $\mathrm{d}\Omega$ with respect to the direction $\mathbf{\Omega}$.

Designate by W the increase of the beam radiation energy in this volume, related to a unit of volume, time (in the vicinity of t), frequency (in the vicinity of ν) and solid angle (with respect to the direction of observation $\mathbf{\Omega}$). Then the quantity

$$W_\nu \, \mathrm{d}A \, \mathrm{d}s \, \mathrm{d}\Omega \, \mathrm{d}\nu \, \mathrm{d}t \tag{9.16}$$

represents the increase of the energy of radiation of a beam concluded in the elementary cylindrical volume $\mathrm{d}A \, \mathrm{d}s$ and propagating within the limits of solid angle $\mathrm{d}\Omega$ with respect to the direction $\mathbf{\Omega}$ for the time interval $\mathrm{d}t$ within the frequency band $\mathrm{d}\nu$.

Equating (9.15) and (9.16), we obtain

$$\frac{\mathrm{d}I_\nu(s, \mathbf{\Omega})}{\mathrm{d}s} = W_\nu. \tag{9.17}$$

Now we can obtain the expression in the explicit form with respect to W_ν using the results obtained in section 9.2. For an absorbing, emitting and scattering medium, quantity W_ν is formed by the components caused by increments and losses of radiation energy:

$$W_\nu = W_{\mathrm{E}} - W_{\mathrm{A}} + W_{\mathrm{IS}} - W_{\mathrm{AS}}. \tag{9.18}$$

The first term on the right-hand side represents the radiation energy increment caused by thermal radiation of a medium and related to the unit of time, volume, solid angle and frequency. If the local thermodynamic equilibrium is established in a medium, then W_{E} will be related to the Planck function and spectral absorption coefficient by relationship (9.3), i.e. $W_{\mathrm{E}} = J_\nu(\mathbf{r})$. The second term represents the radiation energy losses caused by radiation absorption by a medium and related to the unit of time, volume, solid angle and frequency. They can be written as follows:

$$W_A = \gamma_\nu(s) I_\nu(s, \mathbf{\Omega}). \tag{9.19}$$

The third term corresponds to the radiation energy increment caused by radiation falling on a medium from all directions of a spherical space and scattered by a medium in the observation direction. This quantity, like the two previous ones, is related to the unit of time, volume, solid angle and frequency. For purely coherent scattering in the isotropic medium the third term can be presented as

$$W_{\mathrm{IS}} = \frac{1}{4\pi} \sigma_\nu(s) \iint_{4\pi} I_\nu(s, \mathbf{\Omega}') p_\nu(\mathbf{\Omega}') \to \mathbf{\Omega} \, \mathrm{d}\Omega'. \tag{9.20}$$

The last term corresponds to beam energy losses due to radiation scattering by a medium, as a result of which the beams are deflected from the direction $\mathbf{\Omega}$. These losses are also related to the unit of time, volume, solid angle and frequency. They can be written as follows:

$$W_{\mathrm{AS}} = \sigma_\nu(s) I_\nu(s, \mathbf{\Omega}). \tag{9.21}$$

The substitution of the expressions obtained into (9.17) gives the radiative transfer equation in the form of

$$\frac{dI_\nu(s,\mathbf{\Omega})}{ds} + [\gamma_\nu(s) + \sigma_\nu(s)]I_\nu(s,\mathbf{\Omega}) = \gamma_\nu(s)I_{\nu B}[T(s)] + \frac{1}{4\pi}\sigma_\nu(s)\iint_{\Omega'=4\pi} T_\nu(s,\mathbf{\Omega}')p(\mathbf{\Omega}' \to \mathbf{\Omega})\,d\Omega'. \tag{9.22}$$

Most frequently this equation is presented in a more compact form:

$$\frac{1}{\beta_\nu(s)}\frac{dI_\nu(s,\mathbf{\Omega})}{ds} + I_\nu(s,\mathbf{\Omega}) = S_\nu(s), \tag{9.23}$$

where the following designations are used:

$$S_\nu(s) = (1-\omega_\nu)I_{\nu B}[T(s)] + \frac{1}{4\pi}\omega\iint_{\Omega'=4\pi} I_\nu(s,\mathbf{\Omega}')p_\nu(\mathbf{\Omega}' \to \mathbf{\Omega})\,d\Omega', \tag{9.24}$$

$$\beta_\nu(s) = \gamma_\nu(s) + \sigma_\nu(s), \tag{9.25}$$

$$\omega_\nu = \frac{\sigma_\nu(s)}{\gamma_\nu(s) + \sigma_\nu(s)}. \tag{9.26}$$

In these relations $S_\nu(s)$ is called the source function, $\beta_\nu(s)$ the spectral extinction coefficient, $\omega_\nu(s)$ the spectral albedo, which represents the ratio of a scattering coefficient to the extinction coefficient. In studying the transfer processes in gaseous media the spectral albedo is often called the probability of survival of a quantum and is designated as $\Lambda_\nu(s)$. Note once again, that all definitions of medium's parameters presented above are related to the unit volume of substance, rather than to its individual components (for example, the drops of water in a cloud).

Equation (9.22) is the integro-differential partial derivative equation, since the full derivative d/ds contains partial derivatives with respect to spatial coordinates, if it is written in the explicit form for the given coordinate system, and the sought-for intensity $I_\nu(s,\mathbf{\Omega})$ is under the sign of integral in the source function. For this reason the solution of equation (9.22) represents a very complicated problem even for the one-dimensional case. Below we shall consider some important special cases of RTT.

9.4 SPECIAL CASES OF THE RADIATIVE TRANSFER EQUATION

Since obtaining complete solutions of the transfer theory equation for the arbitrary case is rather troublesome, we shall consider some important special cases of RTT, whose solutions are often used in experimental and observational practice (remote sensing, radio-astronomy).

9.4.1 A purely scattering medium

By a purely scattering medium we mean a medium that neither absorbs nor emits thermal radiation, but only scatters electromagnetic radiation, i.e. where $\omega_\nu(s) = 1$, and, hence, $\beta_\nu(s) = \sigma_\nu(s)$. Certainly, in such a case the scattering of the external (with respect to the medium studied) radiation takes place. Natural analogues of such media in the optical band are cloud systems in the terrestrial atmosphere, which consist of small crystals of snow and ice or of volcanic dust particles. The diffuse regime of illumination in such systems has the colloquial designation 'milk'. Important examples of such media can serve for the cloudy atmosphere of Venus and the Martian atmosphere (in the presence of dust-storms). For such media the basic equation (9.23) is simplified:

$$\frac{1}{\beta_\nu(s)}\frac{\mathrm{d}I_\nu(s,\mathbf{\Omega})}{\mathrm{d}s} + I_\nu(s,\mathbf{\Omega}) = \frac{1}{4\pi}\iint_{4\pi} I_\nu(s,\mathbf{\Omega}')p_\nu(\mathbf{\Omega}' \to \mathbf{\Omega})\,\mathrm{d}\Omega'. \tag{9.27}$$

It can easily be seen, however, that the equation for scattering media is still integro-differential and does not have any direct solution. To solve it, one resorts to special methods of solution or to simplifications (Chandrasekhar, 1960; Sobolev, 1963; Ozisik, 1973).

9.4.2 The absorbing and emitting medium

Absorbing and emitting media are characterized by the fact, that they absorb the external radiation passing through them, and emit thermal radiation (the emitted radiation), but do not scatter it, virtually, i.e. $\omega_\nu(s) = 0$ ($\sigma_\nu(s) = 0$). Analogues of such a type of media (for the microwave band) are wide spread in nature. They include: cloud systems (small drops of water, snowflakes and hailstones), clouds of dust, sandy storms, a drop-spray phase on the sea surface, precipitation of various sorts.

For such media the basic equation (9.22) takes the form:

$$\frac{1}{\beta_\nu}\frac{\mathrm{d}I_\nu(s,\mathbf{\Omega})}{\mathrm{d}s} + I_\nu(s,\mathbf{\Omega}) = I_{\nu\mathrm{B}}[T(s)]. \tag{9.28}$$

Unlike the basic equation (9.22) and preceding equation (9.27), the latter equation is purely differential, and its solution can be obtained in the closed form:

$$I_\nu(s,\mathbf{\Omega}) = I_0 \exp(-\beta \mathrm{s}) + \int_0^s I_{\nu\mathrm{B}}[T(s)] \exp(-\beta s')\,\mathrm{d}s', \tag{9.29}$$

where I_0 is the boundary condition, or, otherwise, the intensity of the external (with respect to the medium) radiation at the medium's boundary. For better visualization, we present here the one-dimensional version of the solution of equation (9.29) for a homogeneous medium with respect to the electromagnetic parameters ($\beta_\nu(s) = \gamma_\nu(s) = \text{const}$), but with inhomogeneous heating of the medium ($I_{\nu\mathrm{B}}[T(s)]$). The first term of the solution reflects, how much the external radiation will be absorbed by the medium as the observation point advances in the medium. As

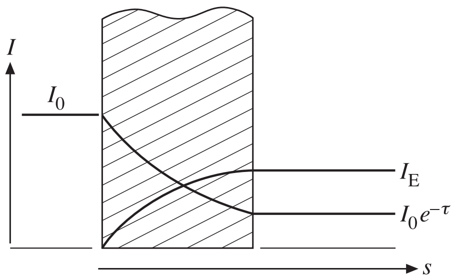

Figure 9.2. Schematic presentation of the radiative transfer passing through the absorbing and emitting one-dimensional layer. I_0 is the external radiation; I_E is the emitted radiation.

should be expected, the external radiation decreases according to the exponential law. The intensity of thermal radiation of the medium is reflected in the second term value and is related to the thermal profile $T(s)$ in a complicated manner (via the Planck function). If the medium is supposed to be not only homogeneous but also isothermal, i.e. $T(s) = T_0 = \text{const}$, then in this case the solution of (9.29) can be reduced to the form:

$$I_\nu(\nu, s) = I_0 \exp(-\beta s) + I_{\nu\mathrm{B}}(\nu, T_0)(1 - \exp(-\beta s)). \tag{9.30}$$

Note that the second term in this expression is none other than the Kirchhoff law. Figure 9.2 shows schematically the relationship between two components formed after external radiation passage through the layer of homogeneous and isothermal medium. The relation obtained is important, since it is often used in experimental practice for various preliminary estimations.

9.4.3 Transparent medium

The non-absorbing, non-emitting and non-scattering medium is called transparent (or diathermal). For such a medium the absorption and scattering coefficients are zero. Substituting $\sigma_\nu(s) = \gamma_\nu(s) = 0$ into equation (9.22), we obtain:

$$\frac{\mathrm{d}I_\nu(s, \mathbf{\Omega})}{\mathrm{d}s} = 0; I_\nu(s, \mathbf{\Omega}) = \text{const}. \tag{9.31}$$

This implies that the radiation intensity in a transparent medium remains constant everywhere in any direction.

9.4.4 The 'cold' layer approximation

The 'cold' layer approximation characterizes the situation, where the external radiation falling on a medium essentially exceeds, in its intensity, the thermal radiation of the medium, which possesses both nonzero absorption coefficient and nonzero scattering coefficient. In other words, the $I_0 \gg I_{\nu\mathrm{B}}[T(s)]$ condition is satisfied. A similar situation is met frequently enough under the natural conditions as well. So, solar radiation in the optical band (under terrestrial conditions) essentially exceeds the thermal radiation of terrestrial media. The power of artificial sources (radio broadcasting, television, communications, radar sources) essentially exceeds thermal radiation of terrestrial media in the microwave band (see Chapter 1). Using this approximation for relation (9.29), we have:

$$I_\nu(s) = I_{\nu_0} \exp\left[-\int_0^s \beta_\nu(z)\,\mathrm{d}z\right]. \tag{9.32}$$

The exponential factor in this expression is often written in another form by introducing the dimensionless quantity

$$\tau(s) = \int_0^s \beta(z)\,\mathrm{d}z. \tag{9.33}$$

The dimensionless quantity τ is called the optical thickness (opacity) of a layer of scattering and absorbing medium having thickness s, and is a function of all values of absorption and scattering coefficients over the spatial scales from 0 to s.

It can easily be seen, that the given relation is none other than the well-known and widely used 'in optics' Bouguer law for absorbing and scattering media (Born and Wolf, 1999; Siegel and Howell, 1972). As far as the microwave sounding problems are concerned, the frameworks of application of the Bouguer law are rather limited, since in this band natural radiations have a compatible order in intensity and, therefore, various constituents should be taken into account when performing measurements under the natural conditions (see Chapters 5 and 12).

9.5 EQUATION OF RADIATIVE TRANSFER FOR THE PLANE-LAYER CASE

As we have noted, the basic equation of radiative transfer is the integro-differential equation, and obtaining its complete solution for the general three-dimensional case is a very complicated problem. However, it is quite useful to trace the formal integration of equation (9.22), so that in some important practical cases it will be possible to obtain results that satisfactorily agree with experimental and observational data. Here we should mention, first of all, the one-dimensional, plane-parallel case. This geometry is widely used in studying the terrestrial atmosphere and terrestrial surfaces over spatial scales where the Earth's curvature does not play a noticeable part.

We shall consider a medium composed of planar layers perpendicular to axis oy, the electromagnetic properties of the medium being constant in each layer. Let s be the length measured in the arbitrary direction $\mathbf{\Omega}$ and θ be the polar angle between the direction $\mathbf{\Omega}$ and positive direction of axis oy (Figure 9.3). The derivative with respect to direction $\mathrm{d}/\mathrm{d}s$ can be expressed in terms of derivatives with respect to the spatial

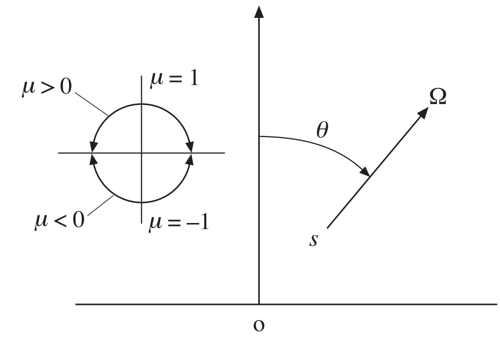

Figure 9.3. The coordinate system for the plane-parallel case. Notation is explained in the text.

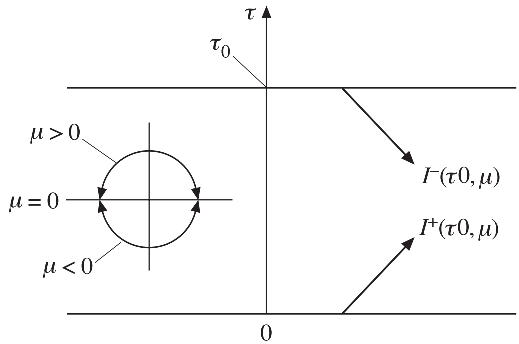

Figure 9.4. The coordinate system for the formal solution of the radiative transfer equation for the plane-parallel case. $I^+(0, \mu)$ and $I^-(\tau_0, \mu)$ are outgoing and incident components of the sought-for radiation. τ_0 is the value of opacity for the upper boundary of a layer.

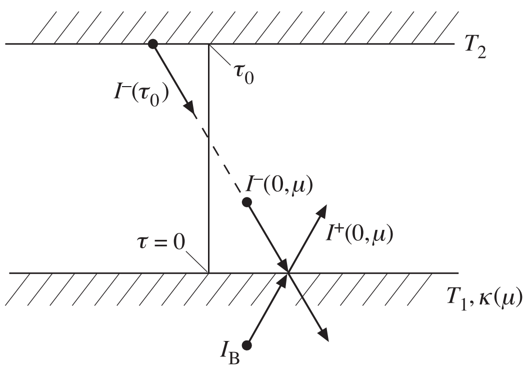

Figure 9.5. The mirror-reflecting, emissive and black boundary conditions for the radiative transfer equation solution. T_2 is the temperature of the upper black-body boundary. T_1 and $\kappa(\mu)$ are the temperature and the emissivity and the lower reflecting boundary. Notation is explained in Figure 9.4.

coordinate y in the form

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\partial}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} = \mu\frac{\partial}{\partial y}, \tag{9.34}$$

where μ is the cosine of angle θ between the radiation transmission direction $\mathbf{\Omega}$ and axis oy, i.e.

$$\mu = \cos\theta, \tag{9.35}$$

and the partial derivatives with respect to x and z for the plane-parallel case are equal to zero. Then the equation of radiative transfer (9.21) takes the following form:

$$\frac{\mu}{\beta_\nu}\frac{\partial I_\nu(y,\mu,\varphi)}{\partial y} + I_\nu(y,\mu,\varphi) = S(y,\mu,\varphi), \tag{9.36}$$

where the source function will be written as

$$S_\nu(y,\mu,\varphi) = (1-\omega_\nu)I_{\nu\mathrm{B}}[T(y)] + \frac{\omega_\nu}{4\pi}\int_{\varphi'=0}^{2\pi}\int_{\mu'=0}^{+1} p(\mu_0)I_\nu(y,\mu,\varphi)\,\mathrm{d}\mu'\,\mathrm{d}\varphi' \tag{9.37}$$

and μ_0 is the cosine of angle between the directions of incident radiation and radiation scattered by a volume element (see equation (9.10)).

Further on, when equation (9.36) is solved mathematically, it is convenient to reduce it to the so-called dimensionless form, making use of the concept of the optical thickness of a layer (9.33). Then equation (9.36) will take the form

$$\mu\frac{\partial I_\nu(\tau,\mu,\varphi)}{\partial \tau} + I_\nu(\tau,\mu,\varphi) = S_\nu(\tau,\mu,\varphi), \tag{9.38}$$

where

$$S_\nu(\tau,\mu,\varphi) = (1-\omega_\nu)I_{\nu\mathrm{B}}[T(\tau)] + \frac{\omega_\nu}{4\pi}\int_{\varphi'=0}^{2\pi}\int_{\mu'=-1}^{+1} p(\mu_0)I_\nu(\tau,\mu,\varphi)\,\mathrm{d}\mu'\,\mathrm{d}\varphi'. \tag{9.39}$$

If the boundary conditions for the equation of radiative transfer are characterized by axial symmetry, then the intensity of radiation in the medium studied does not depend on the azimuthal angle, and equation (9.38) is simplified:

$$\mu\frac{\partial I_\nu(\tau,\mu)}{\partial \tau} + I_\nu(\tau,\mu) - (1-\omega_\nu)I_{\nu\mathrm{B}}[T(\tau)] + \frac{\omega_\nu}{4\pi}\int_{\mu'=1}^{+1} I_\nu(\tau,\mu')\int_{\varphi'=0}^{2\pi} p(\mu_0)\,\mathrm{d}\varphi'\,\mathrm{d}\mu'. \tag{9.40}$$

To fulfil the integration over φ' in this relation and, thus, to essentially simplify the right-hand side of equation (9.40), one resorts to the following approach. The scattering indicatrix $p(\mu_0)$ is expanded over the orthogonal Legendre polynomials (Gradshteyn and Ryzhik, 2000):

$$p(\mu_0) = \sum_{n=0}^{N} a_n P_n(\mu_0);\, a_0 = 1, \tag{9.41}$$

where $P(\mu_0)$ is the Legendre polynomial of the nth order from the argument μ. The physical prerequisites for choosing just such a type of orthogonal expansion are related, first of all, with the fact that the scalar wave equation for systems of particles allows, in the Maxwell theory, the following separation of angular and spatial variables and has particular solutions of the following form (Stratton, 1941):

$$\Psi \sim \begin{matrix} \cos l\varphi \\ \sin l\varphi \end{matrix} \{P_n(\cos\theta)\} Z_{n+1/2}(r). \tag{9.42}$$

In this case the spherical Bessel function $Z_{n+1/2}(r)$ can be presented in the far radiation zone as

$$Z_{n+1/2}(r) \sim [\exp(-\mathrm{j}kr)(kr)^{-1}], \tag{9.43}$$

where k is the wave number.

Thus, using the features of the orthogonal expansion over the Legendre polynomials, the internal integral in the right-hand side of relation (9.40) can be integrated, and the following expression is obtained (see Ozisik (1973) for more details):

$$\int_0^{2\pi} p(\mu_0)\,\mathrm{d}\varphi' = 2\pi \sum_{n=0}^{N} a_n P_n(\mu) P_n(\mu'). \tag{9.44}$$

The substitution of (9.44) into (9.40) represents the equation of radiative transfer in the case of axial symmetry as

$$\mu \frac{\partial I_\nu(\tau,\mu)}{\partial \tau} + I_\nu(\tau,\mu) = (1-\omega) I_{\nu\mathrm{B}}[T(\tau)] + \frac{\omega_\nu}{2} \int_{-1}^{+1} p(\mu,\mu') I_\nu(\tau,\mu')\,\mathrm{d}\mu', \tag{9.45}$$

where

$$p(\mu,\mu') = \sum_{n=0}^{N} a_n P_n(\mu) P_n(\mu'). \tag{9.46}$$

In this case the scattering indicatrix $p(\mu,\mu')$ of a unit volume does not depend on the azimuthal angle. Relation (9.46) has a number of important special cases, which are widely used in observational practice. So, the case of $N = 0$ corresponds to the so-called isotropic scattering, $N = 1$ to the indicatrix of linearly anisotropic scattering, i.e.

$$p(\mu,\mu') = 1 + a_1 \mu\mu', \tag{9.47}$$

and $N = 2$ to the indicatrix of anisotropic scattering of the second order:

$$p(\mu,\mu') = 1 + a_1\mu\mu' + \tfrac{1}{4} a_2 (3\mu^2 - 1)[3(\mu')^2 - 1]. \tag{9.48}$$

The indicatrix of the important case of so-called Rayleigh scattering can be obtained from (9.48) for $a_1 = 0$ and $a_2 = 1/2$, i.e.

$$p(\mu,\mu') = \tfrac{3}{8}[3 - \mu^2 + (3\mu^2 - 1)(\mu')^2]. \tag{9.49}$$

Below we shall transfer to the formal solution of the equation of radiative transfer in the planar case in the presence of axial symmetry (9.45). To solve this equation, one establishes, first of all, the so-called double-flux approximation. For this purpose the unknown intensity $I_\nu(\tau,\mu)$ is separated into two components: the direct (or outgoing) one $I_\nu^+(\tau,\mu)$, $\mu > 0$, and the reverse (or incident) one $I_\nu^- - (\tau,\mu)$, $\mu < 0$ (Figure 9.4). From the viewpoint of experimental practice such a separation is quite lawful and justified since, when a receiving device is at the upper boundary of a layer, we receive radiation formed by the volume of a medium and the escaping (outgoing) from this medium. And when we are at the lower boundary of a medium, we receive radiation falling (incident) from the medium's volume on a receiving device. The separation of the intensity into two components in performing particular measurements is clear enough and, therefore, no special explanations are usually made in describing the experiments. In this approach the equations for outgoing and incident components of the sought-for radiation and the corresponding boundary conditions will be as follows:

$$\mu \frac{\partial I_\nu^+(\tau,\mu)}{\partial \tau} + I_\nu^\pm(\tau,\mu) = S(\tau,\mu), \tag{9.50}$$

$$I_\nu^+(\tau,\mu)|_{\tau=0} = I_\nu^+(0,\mu); 0 < \mu \le 1, \tag{9.51}$$

$$I_\nu^-(\tau,\mu)|_{\tau=\tau_0} = I_\nu^-(\tau_0,\mu); -1 \le \mu < 0. \tag{9.52}$$

These equations are not independent, however, but represent an interdependent system, since they contain the source function, which can be written in this representation as:

$$S_\nu(\tau,\mu) = (1-\omega_\nu) I_{\nu \mathrm{B}}[T(\tau)] + \frac{\omega_\nu}{2}\left[\int_0^1 p(\mu,\mu') I_\nu^+(\tau,\mu')\, \mathrm{d}'\mu' + \int_{-1}^0 p(\mu,\mu') I_\nu^-(\tau,\mu')\, \mathrm{d}\mu'\right]. \tag{9.53}$$

The formal solution of equations (9.50) can be obtained by means of the well-known integrating multiplier method. For an outgoing flux we have:

$$I_\nu^+(\tau,\mu) = I_\nu^+(0,\mu)\exp\left(-(\tau/\mu)\right) + \frac{1}{\mu}\int_0^\tau S(\tau',\mu)\exp\left(-(\tau-\tau')/\mu\right)\mathrm{d}\tau' \tag{9.54}$$

for $\mu > 0$.

For an incident flux after traditional replacement of μ by $-\mu$ the solution has the form:

$$I_\tau^-(\tau,-\mu) = I_\tau^-(\tau_0,\mu)\exp\left(-(\tau_0-\tau)/\mu\right) + \frac{1}{\mu}\int_\tau^{\tau_0} S_\nu(\tau,-\mu)\exp\left(-(\tau'-\tau)/\mu\right)\mathrm{d}\tau'. \tag{9.55}$$

In these relations, for example (9.54), the first term in the right-hand side represents, in the explicit form, the contribution of radiation from the boundary surface with $\tau = 0$, which has attenuated on passage through a medium to depth τ without scattering. The second term represents the contribution of the source function within the range of values from $\tau = 0$ to τ into the intensity of radiation at depth τ. The terms of relation (9.55) have similar physical sense (with correction for geometry). The formal expressions (9.54) and (9.55) are not solutions in the true sense, since in the general case the source function and the intensities at the boundaries depend on the unknown intensity of radiation emitted by the medium. And, therefore, they cannot be directly used as initial expressions in solving the problem under consideration. Below we shall demonstrate how the problem can be solved up to the final result for a series of important practical cases.

9.6 BOUNDARY CONDITIONS

In section 9.5 the formal values of functions $I_\tau^+(\tau, \mu), \mu > 0$ and $I_\nu^-(\tau, \mu), \mu < 0$ have been used as boundary conditions at the boundaries of $\tau = 0$ and $\tau = \tau_0$, respectively. In this section we shall present explicit expressions for these boundary conditions in the case of transparent and non-transparent boundary surfaces being diffuse and mirroring reflectors.

9.6.1 The transparent boundaries

If the boundary surfaces $\tau = 0$ and $\tau = \tau_0$ are transparent, and the adjacent surrounding space is a vacuum (i.e. it does not interact with radiation), then the boundary conditions for the incident (from outside) radiation in the case of axial symmetry can be written as

$$I_\nu^+(0, \mu) = f_{1\nu}(\mu); \mu > 0, \tag{9.56}$$

$$I_\nu^+(\tau_0, \mu) = f_{2\nu}(\mu); \mu < 0, \tag{9.57}$$

where $f_{1\nu}(\mu)$ and $f_{2\nu}(\mu)$ are specified functions of parameter μ. If the incident radiation, falling on a studied layer from outside, is constant, formulas (9.56) and (9.57) are simplified to the form

$$I_\nu^+(0) = f_{1\nu}; \mu > 0, \tag{9.58}$$

$$I_\nu^-(\tau_0) = f_{2\nu}; \mu < 0, \tag{9.59}$$

where $f_{1\nu}$ and $f_{2\nu}$ are constants.

Typical examples of such kinds of boundary conditions are solar radiation falling on the upper boundary of the terrestrial atmosphere and other extra-terrestrial sources of radio-emission of galactic and extra-galactic origin.

9.6.2 The black boundaries

If both boundary surfaces $\tau = 0$ and $\tau = \tau_0$ are black (i.e. fully absorbing the incident radiation) and are maintained at constant temperatures T_1 and T_2, respectively, then the spectral intensity of radiation emitted by these surfaces is described by the Planck function at the surface temperature (see Chapter 6). Then the boundary conditions can be written as:

$$I_\nu^+(0) = I_{\nu\mathrm{B}}(T_1), \tag{9.60}$$

$$I_\nu^-(\tau_0) = I_{\nu\mathrm{B}}(T_2), \tag{9.61}$$

where $I_{\nu\mathrm{B}}(T)$ is the Planck function, whose value does not depend on the direction.

A typical natural example of such a type of boundary (for the microwave band) is the upper boundary (conventional, in a certain sense) of the terrestrial atmosphere, on which falls the black-body radiation of the relic background of the universe with brightness temperature of $T_2 = 2.7\,\mathrm{K}$.

9.6.3 Mirror-reflecting and black boundaries

In the microwave band, as we have already argued (see Chapter 7), some part of terrestrial surfaces can be considered in the approximation of mirror-reflecting media with a power reflection coefficient distinct from unity. Thus, such a type of boundary on the one hand will be the source of thermal radiation, and, on the other hand, will reflect the incident radiation falling on it from the studied medium.

Now we consider the situation with the boundary conditions, which is quite close to real situations in experimental practice when studying the terrestrial atmosphere. In such a case, the upper boundary represents a black body with a brightness temperature of 2.7 K (see relation (9.61)). The radiation at the lower boundary will be formed from the thermal radiation of the surface with emissivity $\kappa_1(\mu)$ and temperature T_1 and from re-reflected radiation with the Fresnel power coefficient $|R(\mu)|^2$, which was formed at the lower boundary of a studied layer. In virtue of the fact that the emissive and reflective properties of the surface depend on the polarization of radiation received by a receiving system, the outgoing flux under these conditions will also possess polarization properties, though distinct from the polarization properties of the surface itself. Corresponding examples will be considered below.

So, for the conditions at the upper black-body boundary and at the lower reflecting boundary (Figure 9.5) the unknown boundary conditions can be presented in the form:

$$I^-(\tau_0) = I_{\nu\mathrm{B}}(T_2), \tag{9.62}$$

$$I_\nu^+(0,\mu) = \kappa(\mu) I_{\nu\mathrm{B}}(T_1) + |R_\nu(\mu)|^2 I_\nu^-(0,\mu). \tag{9.63}$$

We shall repeatedly use the boundary conditions of type (9.62) and (9.63) hereafter.

9.7 RADIATIVE TRANSFER IN THE EMITTING AND ABSORBING MEDIUM

One of the most important particular cases in the radiative transfer theory is the approximation of emitting and absorbing medium (without scattering, i.e. $\omega = 0$). This approximation is especially widely used in the microwave band. For the conditions of purely gaseous terrestrial atmosphere, without the presence of hydrometeors, the relation $\omega = 0$ is met accurately. But even in the presence of hydrometeors at wavelengths greater than 1 cm this condition is met to a good accuracy (see Chapter 10). For this reason, we shall present in this section the explicit expressions of radiation intensity for some particular observational schemes.

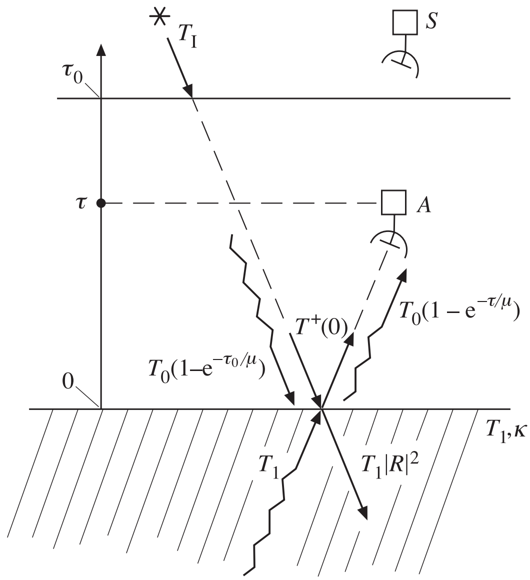

Figure 9.6. The measurement scheme for recording the outgoing radiation by the aircarrier (A) (aircraft) inside the atmosphere and by a satellite (S) outside the atmosphere. Notation is explained in the text.

Figure 9.6 presents the measurement scheme for recording the outgoing radiation in the conventional electrodynamically homogeneous, non-isothermal atmosphere over the solid (or liquid) surface. The recording microwave device is installed either on an aircraft inside the atmosphere at the given altitude (the dimensionless coordinate τ), or on a satellite outside the atmosphere. We shall suppose that outside the conventional atmosphere the attenuation in a medium is absent; therefore, for a satellite version the altitude coordinate can be $\tau = \tau_0$.

As we have already noted, the Rayleigh–Jeans approximation is valid in the microwave band, and, hence, we can proceed to presentation of the solution of (9.54) and (9.55) in the form of brightness temperatures. Then the source function will be equal to $S(\tau) = T_0(\tau)$ and, thereby, it reflects the non-isothermal character of the atmosphere.

Thus, the complete solution for outgoing radiation, which is recorded at the dimensionless altitude $\tau(h)$, will be equal to:

$$T_B^+(\tau,\mu) = T_B^+(0,\mu)\exp\left(-(\tau/\mu)\right) + \frac{1}{\mu}\int_0^{\tau(h)} T_0(\tau')\exp\left(-(\tau-\tau')/\mu\right)d\tau'; \mu > 0. \tag{9.64}$$

The boundary condition at the lower boundary will be formed from two components – the thermal radiation of the surface and the incident radiation from the atmosphere, re-reflected by this surface:

$$T_{\mathrm{B}}^{+}(0,\mu) = \kappa(\mu)T_2 + |R(\mu)|^2 T_{\mathrm{B}}^{-}(0,-\mu). \tag{9.65}$$

In its turn, the incident radiation, falling from the atmosphere to the lower boundary, will also consist of two components – the external radiation (the radiation of illumination) T_{I} fallen on the upper atmosphere's boundary and attenuated in the atmosphere medium, and the thermal radiation of the atmosphere formed inside the atmosphere. The complete solution for incident radiation at the lower boundary will take the form:

$$T_{\mathrm{B}}^{-}(0,-\mu) = T_{\mathrm{I}}\exp(-(\tau_0/\mu)) + \frac{1}{\mu}\int_0^{\tau_0} T_0(\tau')\exp(-(\tau'/\mu))\,\mathrm{d}\tau';\ \mu > 0. \tag{9.66}$$

For greater physical clarity we assume the atmosphere to be isothermal, i.e. $T_0(\tau) = T_0$. Note that the isothermal approximation for atmospheric problems should be used with great caution, since it is known from the thermohydrodynamics that such atmospheres are dynamically unstable under gravity conditions.

So, the expressions for the isothermal atmosphere will be as follows:

$$T_{\mathrm{B}}^{+}(\tau,\mu) = T_{\mathrm{B}}^{+}(0,\mu)\exp(-(\tau/\mu)) + T_0[1-\exp(-(\tau/\mu))], \tag{9.67}$$

$$T_{\mathrm{B}}^{+}(0,\mu) = \kappa(\mu)T_2 + |R(\mu)|^2[T_{\mathrm{I}}\exp(-(\tau_0/\mu)) + T_0(1-\exp(-(\tau_0/\mu)))]. \tag{9.68}$$

Note that the outgoing recorded radiation consists of two components – the contribution of the atmosphere itself and the contribution from the surface and external radiation. As we have already noted, all these components possess identical statistical properties and cannot be distinguished by this criterion. So, it is necessary to use the polarization features of total radiation to separate various components. Note that the last term in these expressions is none other than the numerical expression of Kirchhoff's law.

The obtained expressions for outgoing radiation (9.64) and (9.67) are widely used in various modifications in experimental microwave remote sensing practice.

9.8 FEATURES OF RADIATION OF A HALF-SPACE WITH THE SEMI-TRANSPARENT ATMOSPHERE

In this section we shall consider in more detail the features of radiation of the surface–atmosphere system and also will present the observational techniques that are used in studying the electromagnetic properties of the atmosphere.

For physical clarity we shall consider the simplest isothermal version, where both the atmosphere and the surface are at the same thermodynamic temperature T_0, and external radiation is absent, i.e. $T_1 = 0$. Using relation (9.67), we obtain in this case the expression for the outgoing flux intensity at the upper boundary for observation into the nadir ($\mu = 1$) in the following form:

$$T_{\mathrm{B}}(\tau_0, 0) = T_0[1 - |R(0)|^2\exp(-2\tau_0)]. \tag{9.69}$$

It follows from this relation that the emissivity of the surface–atmosphere system κ_{SA} will be equal to:

$$\kappa_{SA}(\tau_0, 0) = 1 - |R(0)|^2 \exp(-2\tau_0). \tag{9.70}$$

Recall that in the absence of the atmosphere the surface emissivity κ_S will be

$$\kappa_S(\tau_0, 0) = 1 - |R(0)|^2. \tag{9.71}$$

Note that the doubled value of the optical thickness (path) in the exponent is physically related to the fact that three types of radiation make contribution to the total radiation in a statistically independent manner, namely: (1) the thermal outgoing radiation of the atmosphere, (2) the re-reflected incident radiation of the same atmosphere, and (3) the thermal radiation of the surface.

9.8.1 Brightness contrast

Note that in experimental practice are brightness contrast is important in studying complex geophysical objects. Suppose we are interested in the brightness contrast at observation of the surface–atmosphere system and surface only. In this case the expression for the brightness contrast can be presented as:

$$\Delta T_B(\tau_0, 0) = T_{BSA}(\tau_0, 0) - T_{BS}(\tau_0, 0) = T_0|R(0)|^2(1 - \exp(-2\tau_0)), \tag{9.72}$$

and the emissivity contrast can be written as:

$$\Delta\kappa = \kappa_{SA} - \kappa_S = |R(0)|^2(1 - \exp(-2\tau_0)). \tag{9.73}$$

If the homogeneous atmosphere possesses semi-transparent properties, i.e. $t < 1$, then in this case, expanding the exponential function into a series and retaining the first two terms, we shall have

$$\Delta T_B(\tau_0, 0) \cong T_0|R(0)|^2 2\gamma h, \tag{9.74}$$

$$\Delta\kappa(\tau_0, 0) \cong |R(0)|^2 2\gamma h. \tag{9.75}$$

The relations obtained give rise to some important consequences, which are widely used in experimental practice. First, the brightness contrast is always positive in the presence of absorbing and emitting atmosphere. However, as we shall see below, the situation can drastically change when scattering is present in the atmospheric formations. Second, the value of contrast for a semi-transparent atmosphere is proportional to the electrodynamic properties of the atmosphere medium, and, knowing the values of temperature and atmosphere's height from accompanying measurements, we can obtain the value of attenuation in the atmosphere substance. Third, as the value of the optical path of the atmosphere increases (via increasing the height or attenuation in the atmosphere), the emissive properties of a system will tend to the properties of black-body radiation (Figure 9.7). In this situation the information on the surface and the electrodynamic properties of the atmosphere will be completely 'blocked' (see Chapter 6).

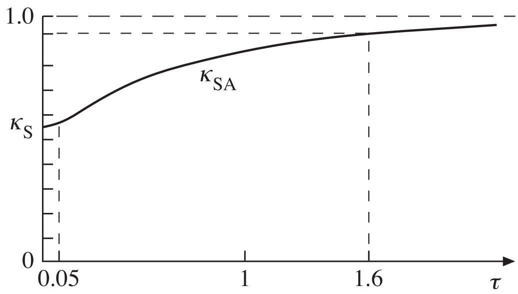

Figure 9.7. Emissivity of the atmosphere–surface system as a function of the atmospheric optical thickness. The symbol κ_S presents the surface emissivity; κ_{SA} is the atmosphere–surface system emissivity.

By virtue of this circumstance, in experimental practice the atmosphere is subdivided (fairly conventionally, of course) into three types: the transparent atmosphere with $\tau < 0.05$ (and, accordingly, with the emissivity contrast $\Delta\kappa < 0.1$); the non-transparent atmosphere with system emissivity $\kappa_{SA} > 0.95$ and accordingly, $\tau \geq 1.6$, and the semi-transparent atmosphere with optical path values in the range of $0.05 < t < 1.6$ (Figure 9.7). It can easily be seen that the measurements in the semi-transparent atmosphere will be most informative for remote sensing, as we shall see below.

9.8.2 Angular measurements

By virtue of the fact, that the emitting half-space possesses polarization properties (see Chapter 7), these properties will reveal themselves (in a rather peculiar manner, however, as we shall see later) in measurements of the surface–atmosphere system as well.

So, for the isothermal, planar surface–atmosphere system under observation at angle θ we obtain from relations (9.67) and (9.68) the following value for the outgoing radiation intensity:

$$\kappa_{SA_i}(\mu) = 1 - |R_i(\mu)|^2 \exp(-2\tau_0/\mu). \tag{9.76}$$

where $i = H, V$ are the horizontal and vertical components of an outgoing flux, respectively, and $\mu = \cos\theta$.

As we already know, the emissivity of the planar half-space (both components) tends to zero as the observation angle tends to 90°, and, accordingly, the Fresnel coefficient tends to unity. However, it can easily be seen from (9.73), that for $\theta \to 90^\circ$ and, passing to the other complementary angle $\alpha = 90^\circ - \theta$, the limiting value of κ_{SA} can be written as:

$$\kappa_{SA_i}(\alpha) \cong 1 - \exp(-2\tau_0/\alpha) \to 1 \tag{9.77}$$

for any values of the optical path of the atmosphere. In other words, the radiation of the surface–atmosphere system represents black-body radiation at grazing observation angles (Figure 9.8). In such a case measurement of the surface's properties and

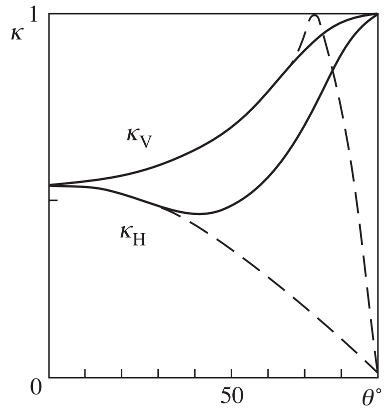

Figure 9.8. Polarization dependences of the atmosphere–surface system (solid lines) and of the surface (dashed lines).

of the electrodynamic properties of the atmosphere is impossible. The physical sense of such a paradoxical (at first sight) situation is related to the initial specifying of a plane-layered model of the atmosphere. In real atmosphere investigation practice it is necessary, of course, to take into account the sphericity of the atmosphere, which eliminates the paradoxical effect mentioned. It is important to note that such an observation mode, which is called the limb method of studying the atmosphere, has been widely disseminated recently, since it makes it possible to record and study in detail very fine features of radiation both of the terrestrial atmosphere (Hartmann *et al.*, 1996; Masuko *et al.*, 2000), and of the atmospheres of planets (Mars, in particular) as well.

9.8.3 The oblique section method

The features of thermal signal transmission on oblique tracks allow us to use some measurement techniques, very useful in observations, which are generically called the oblique section method. This method was proposed and developed in radioastronomical practice. Now it is used, with various modifications, in remote observation as well.

The observational scheme of the oblique section method is as follows. The instrument is situated at the lower boundary of the atmosphere layer and records the intensity of an incident flux (9.66). If there is a strong thermal source with brightness temperature T_{BS} outside the atmosphere, then the total intensity of an incident flux can be written as

$$T_B^-(\mu) = T_{BS} \exp(-\tau_0/\mu) + T_0(1 - \exp(-\tau_0/\mu)), \tag{9.78}$$

(for the sake of convenience we have replaced μ with $-\mu$).

Here the first term describes the received radiation from the extraneous (external) source, and the second term corresponds to the contribution of the thermal radiation of the atmosphere. This makes it possible to follow one of two experimental techniques: either to study source intensity variations as the source passes across the sky (for 'regular transmission'), or to use forced scanning over the observation angle at reception of thermal radiation of the atmosphere. Each of these techniques possesses both positive and negative features in their direct use in observational practice (Haroules and Brown, 1968; Gorelik *et al.*, 1975).

A principal feature of the oblique section method is the circumstance that the variation of the angular dependence of a received signal in various modifications is identical, namely, as a secant of the observation angle. This makes it possible to essentially simplify the measurements, that is, to avoid absolute radiothermal measurements and pass to the mode of relative measurements. The latter approach is, of course, much easier and more reliable both methodologically and in respect of the technological implementation.

To better understand the essence of the oblique section method we rewrite the expression for an incident flux in the following form:

$$\frac{T_{\mathrm{B}}^{-} - T_0}{T_{\mathrm{BS}} - T_0} = \exp(-\tau_0 \sec\theta). \tag{9.79}$$

Differentiating over the parameter $\sec\theta$, we obtain from (9.76) the following expression:

$$-\frac{1}{T_{\mathrm{B}} - T_0}\frac{\mathrm{d}[T_{\mathrm{B}}(\theta) - T_0]}{\mathrm{d}(\sec\theta)} = \tau_0 \exp(-\tau_0 \sec\theta), \tag{9.80}$$

and, substituting here the expression for an exponential function from (9.79), we find the relation sought:

$$\tau_0 = -\left(\frac{1}{T_{\mathrm{B}} - T_0}\right)\frac{\mathrm{d}(T_{\mathrm{B}} - T_0)}{\mathrm{d}(\sec\theta)}. \tag{9.81}$$

And, passing to finite differences, we shall have

$$\frac{\Delta[T_{\mathrm{B}}(\theta) - T_0]}{T_{\mathrm{B}}(\theta) - T_0} = -\tau_0\,\Delta(\sec\theta). \tag{9.82}$$

Thus, quantity τ_0 can be determined from the relative measurements of variations of the external signal intensity as a function of $\sec\theta$.

A similar approach can also be used in measuring the thermal radiation of the atmosphere (the second term in expression (9.78)). Performing a similar operation, we shall obtain the expression for the optical path in the form:

$$\tau_0 = \frac{1}{T_{\mathrm{B}} - T_0}\frac{\mathrm{d}(T_{\mathrm{B}} - T_0)}{\mathrm{d}(\sec\theta)}. \tag{9.83}$$

In other words, the optical path value can be obtained as a tangent of the angle of a slope of the plot of relative variations of thermal radiation versus $\sec\theta$. As an indicative example, we shall present the data of measurements of thermal radiation of a cloudless atmosphere (for clear weather conditions), carried out simultaneously at two frequencies: 19 GHz and 35 GHz (Figure 9.9) (Haroules and Brown, 1968). It follows from the measurement data, that under the meteorological

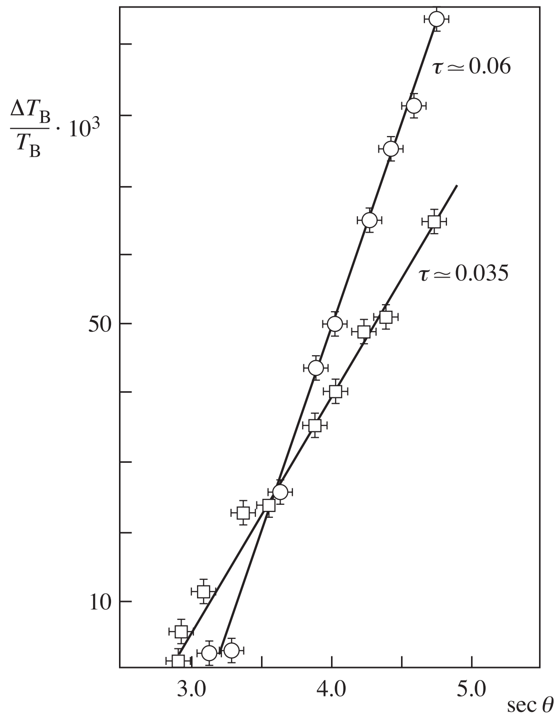

Figure 9.9. Measured atmospheric opacity at 19 (squares) and 35 GHz (circles) under clear weather conditions (temperature, 4°C; pressure 1000.9 mbar; water vapour 2.4 g/m^3) (Haroules and Brown, 1968).

conditions studied the opacity of the terrestrial atmosphere at the frequency of 19 GHz was found to be 0.035, and at the frequency of 35 GHz, 0.06. It follows from this result that under the meteorological conditions studied the state of the atmosphere can be characterized as a transparent atmosphere.

A final objective of this kind of experiment is obtaining spectral characteristics of absorption of the atmosphere within a wide wavelength range, which, in its turn, determines the physicochemical and aggregate composition of the atmosphere (see Chapters 10 and 11).

9.9 RADIATIVE TRANSFER IN THE EMITTING, ABSORBING AND SCATTERING MEDIUM

To evaluate the contribution of scattering to radiative transfer, we consider a model situation where we shall take into account the full scattering losses, but disregard the contribution of rescattering (see section 9.2). In other words, the source function will be taken into account as it is written in relation (9.53); however, we shall still disregard the contribution of the integral, i.e. we let $p(\mu_0) = 0$. This model approach makes it possible to evaluate the upper boundary of possible losses during radiative transfer in a scattering medium.

The values of scattering albedo for particles in the terrestrial atmosphere for the millimetre and centimetre bands vary within very wide limits, from 0.1 to 0.7. In the optical band, however, the albedo in cloudy systems can reach the values of 0.99 and greater.

Taking into account the model approach conditions, we obtain from relations (9.54) and boundary conditions (9.63) the expression for an outgoing flux at the upper boundary of the atmosphere in the form of

$$T_B^+(\tau_0, \mu) = \kappa T_2 \exp(-\tau_0/\mu) + (1 + |R(\mu)|^2 \exp(-\tau_0/\mu))(1-\omega)T_0(1 - \exp -\tau_0/\mu)). \tag{9.84}$$

It follows from the expression obtained that, as $(\tau_0/\mu) \to \infty$, the limiting value of intensity tends to $(1-\omega)T_0$. In other words, under such conditions the surface–atmosphere system emits as a black-body emitter, but with essentially lower effective thermodynamic temperature. In this case one sometimes says, that the 'cooling' of a medium occurs because of 'the inner radiative scattering losses'. The incident radiation, falling on a volume element and scattered by it, will be subjected to further multiple scattering on other medium's elements and, eventually, will dissipate in a medium. The processes of multiple scattering in scattering media are, certainly, very complicated and represent a subject of independent investigation (Marchuk, 1976; Ishimaru, 1978, 1991).

As we have already noted, of importance in the observational practice are background contrasts – the difference between radiation of the surface–atmosphere system and radiation of the surface only. Making easy transformations using (9.84), we obtain the expression the radiothermal contrast at observation of the nadir:

$$\Delta T_B(\mu = 1) = T_0|R|^2(1 - \exp(-2\tau_0))\left[1 - \frac{\omega}{|R|^2}\frac{1 + |R|^2\exp(-\tau_0)}{1 + \exp(-\tau_0)}\right]. \tag{9.85}$$

And, assuming the atmosphere to be transparent ($\tau \ll 1$), we simplify expression (9.85) to the form:

$$\Delta T_B \cong T_0|R|^2 2\tau_0\left[1 - \frac{\omega}{2}\frac{1 + |R|^2}{|R|^2}\right]. \tag{9.86}$$

Let us analyse the relations obtained. First, it should be noted at once that the contrast in the presence of scattering atmosphere (unlike non-scattering atmosphere) can have both positive and negative sign. Second, the contrast depends, in a rather complicated manner, not only on the properties of the atmosphere itself, but on the emissive properties of the surface. If the surface is rather 'cold' in the radio-thermal sense, i.e. $\kappa \to 0$, then the value of contrast is positive and can be presented as

$$\Delta T_B = T_0(1 - \exp(-2\tau_0))(1-\omega) \cong T_0 2\tau_0(1-\omega). \tag{9.87}$$

In the opposite case, for 'warm' 'black-body' surfaces ($\kappa \to 1$), the situation is reversed: the contrast is negative, and its value can be written as:

$$\Delta T_{\mathrm{B}} = -T_0\omega(1 - \exp(-2\tau_0)) \cong -2T_0\omega\tau_0. \tag{9.88}$$

From relation (9.86) it can also easily be seen that at a certain value of surface emissivity the value of contrast will be zero. This value of κ can be estimated as

$$\kappa \cong \frac{2(1-\omega)}{2-\omega}. \tag{9.89}$$

It can be seen from this relation that, for example, for a cloud with albedo $\omega = 0.7$ the surface emissivity, for which the effect of presence of a cloud is absent, will be equal to $\kappa = 0.5$.

The physical meaning of the obtained results is rather transparent. In the case of 'warm' surfaces, the scattering cloud does not compensate in a full measure for those scattering losses of radiation, outgoing from the surface, which are introduced by the cloud itself. In the case of 'cold' surfaces the situation is the reverse – the cloud not only fully compensates scattering losses, but, in addition, makes its own contribution into the total radiation, thus providing a positive contrast.

The importance of the model situation, considered above, lies in the fact that this simple example clearly demonstrates the important fact that the scattering can drastically change the whole radiation energetics in an emitting and scattering system.

9.10 RADIATION OF THE INHOMOGENEOUS AND NON-ISOTHERMAL HALF-SPACE

Making use of the formal solution of the basic transfer equation (9.54) and (9.55), we obtain the explicit expression for radiation intensity in another important case – for a medium with stratified electromagnetic and thermal parameters. We mean the non-isothermal half-space with inhomogeneous electromagnetic properties. The natural analogues of such media are widespread: they include both inhomogeneous soils and grounds with complicated moisture and temperature profiles, inhomogeneous vegetation with a complicated internal thermal regime, the non-isothermal surface microscopic layer of the ocean, inhomogeneous rocks, and surface layers of the Moon, Mars and other planets.

So, we shall consider the absorbing and emitting medium (without scattering, $\omega = 0$) with arbitrary profiles of electromagnetic properties $\gamma(z)$ and temperature $T_0(z)$ (here z denotes the layer depth from the surface). We shall consider, for convenience, the solution of (9.52) for an incident flux at the lower boundary of a layer:

$$I_\nu^-(0,\mu) = I^-(\tau_0,\mu)\exp(-\tau_0/\mu) + \frac{1}{\mu}\int_0^{\tau_0} S(\tau',\mu)\exp(-\tau'/\mu)\,\mathrm{d}\tau'. \tag{9.90}$$

Since we consider the half-space, we shall tend $\tau_0 \to \infty$ and pass from the dimensionless optical path to the dimensional layer depth z. Then we use the presentation of intensity in terms of the brightness temperature. And, finally, after some transformations we shall have the expression for the so-called effective temperature T_{ef} of the non-isothermal and inhomogeneous half-space, measured from the internal side of a layer, in the form:

$$T_{\text{ef}}(\theta) = \int_0^\infty T(z)\gamma(z)\frac{1}{\cos\theta}\exp\left\{-\frac{1}{\cos\theta}\int_0^z \gamma(z')\,\mathrm{d}z'\right\}\mathrm{d}z. \tag{9.91}$$

Since in considering half-spaces the coordinate z in the positive direction is usually pointed to the depth of a layer, we change the coordinate system from what was accepted earlier in studying the atmospheres to the opposite one. Note also that angle θ in expression (9.91) represents an internal angle in a medium (rather than the external observation angle θ_0 related to θ by Snell's law). A fairly complicated functional dependence of the electrodynamic properties of a medium under the sign of integral is explained by the following circumstance. Any unit layer in a medium absorbs that radiation which passes through it from underlying layers and, at the same time, it emits thermal energy that will be partially absorbed by overlying layers. The intensity expressed in (9.91) is formed directly under the half-space boundary; and, finally, that energy will escape into free space, which is proportional to the following value:

$$T_{\text{B}i}(\theta_0) = [1 - |R_i(\theta_0)|^2]\,T_{\text{ef}}(\theta), \tag{9.92}$$

where $i = H, V$ (horizontal and vertical polarizations).

It is important to note that the internal radiation of a medium, described in terms of effective temperature, does not possess polarization properties. The radiation acquires these properties only after intersecting the planar boundary.

By virtue of the aforementioned specificity of thermal radiation formation, it can easily be concluded from expression (9.91) that the depth of the layer at which the basic part of emitted energy can be formed has a quite finite value. We have already made this estimation for moist soils in section 8.8. For this purpose we shall consider the isothermal medium with homogeneous parameters and variable lower limit (the depth). Then at observation into the nadir we obtain the expression for the brightness temperature in the form:

$$T_{\text{B}}(z) = [1 - |R(0)|^2]\,T_0(1 - \exp(-\gamma z)). \tag{9.93}$$

It directly follows from this result, that the effective depth z_{ef} of homogeneous space, which forms 90% of radiation intensity intensity (the so-called skin layer of radiation), equals the following value:

$$z_{\text{ef}} = \frac{2.3}{\gamma} = 0.18\frac{\lambda}{\sqrt{\dfrac{\varepsilon_1}{2}\left(\sqrt{1 + \mathrm{tg}^2\delta} - 1\right)}}, \tag{9.94}$$

where ε_1 and tgδ are electrical parameters of an emitting medium.

If the medium is transparent, i.e. $\mathrm{tg}\delta \ll 1$, then the above expression is simplified:

$$z_{\mathrm{ef}} \cong 0.36 \frac{\lambda}{\sqrt{\varepsilon_1}\,\mathrm{tg}\delta}. \tag{9.95}$$

It can easily be seen from this relation, that for terrestrial media the values of effective depths vary within very wide limits. So, for the glacial ice, whose electrical parameters are $\varepsilon_1 = 3$ and $\mathrm{tg}\delta \approx 0.001$, for the decimetre wavelength band (30 cm for instance) the effective depth will be 63 m. In the same wavelength band for fresh water ($t = 0^\circ\mathrm{C}$) ($\varepsilon_1 = 80$ and $\mathrm{tg}\delta \approx 0.04$) the effective depth will be about 30 cm, whereas for salt water, under the same conditions, $z_{\mathrm{ef}} \approx 1.3$ cm. In the millimetre band (8 mm) the skin layer of radiation for an aqueous medium will be 1 mm only.

If the emitting medium is highly inhomogeneous in electrical and temperature parameters, then the direct estimation of a skin layer from relations (9.94) and (9.95) is unacceptable, strictly speaking, since the picture of internal radiation can be very complicated (see section 7.7.2).

It is interesting to note that if we transfer to the isothermal case ($T_0(z) = T_0$), then the complex integral (9.91) for effective temperature transforms to the value T_0 regardless of the profile of electrodynamic properties of a medium. In other words, the semi-infinite isothermal medium represents a black-body emitter for any values of electrodynamic properties and profiles.

Comparing the expression obtained with relation (7.100), we can easily see their full identity. However, at the same time, both the limits of applicability of radiative transfer theory and clear limitations in using this theory become obvious (see section 7.7.3). This is due to the fact that in the presence of electrical losses in the studied medium ($\mathrm{tg}\delta \neq 0$) Snell's law should be used, strictly speaking, in the complex form and, hence, the value of angle θ inside the medium will also be complex. Thereby expression (9.91) loses its physical sense as radiation intensity. Thus, strictly speaking, the results of radiative transfer theory are applicable for transparent media only. However, some special investigations, carried out beyond the radiative transfer theory framework (Shulgina, 1975; Sharkov, 1978; Klepikov and Sharkov, 1983), have shown, in fact, that the situation is not so dramatic. The contribution of absorbing properties of a medium to refractive characteristics of a medium is quite small (see section 7.7.3). And, therefore, the transfer theory results can be successfully used for media with considerable absorption (such as sea water) as well.

The aforementioned formulas (9.91) and (9.92) are widely used in analysing the emissive properties of inhomogeneous and non-isothermal media, both for remote sensing applications and in radio-astronomy. So, it was from radio-astronomical (remote) observations, using the transfer theory results, that the features of the thermal regime of the subsurface layers of the Moon were revealed and, it was by means of Krotikov's relations (8.52) that the physicochemical properties of surface and subsurface layers of the Moon were first determined (Troitskii, 1954, 1967; Tikhonova and Troitskii, 1970). These results served as a basis for developing the modules designed for landing on the lunar surface. Similar investigations have been subsequently carried out for the Martian surface as well.

9.11 APPROXIMATE METHODS FOR SOLUTION OF THE COMPLETE TRANSFER EQUATION

The mathematical difficulties that arise in solving the complete integro-differential equation of the transfer theory (9.23) have resulted in the appearance of a series of approximate approaches and methods for solution of the radiative transfer equation. At present, the approximate methods of solution of the radiative transfer equation form an independent mathematical discipline. Here it should be noted that quite different (initially) physical prerequisites are laid down in various approaches, and, therefore, the spheres of applicability of these methods are very different from each other. As a result, the matching of solutions of various approximate methods among themselves, sometimes represents, a very complicated problem in itself. So, in the approximations of thin-optical and thick-optical layers (the latter is also called the diffusive approximation, or the Rosseland approximation) simplifications are used that follow from the corresponding limiting value of the medium's thickness. In Eddington's and Schuster-Schwarzchild's approximations the simplifications are related to the introduction of some special assumptions on the angular distribution of radiation intensity. In the method of exponential approximation of a core the integro-exponential functions in the formal solution are replaced by the exponents. The spherical harmonics method and the Gaussian quadratures method are the most well-developed techniques allowing us to obtain high-order approximations using fairly simple procedures.

In this paragraph we shall describe two of the aforementioned approximate methods for solution of the radiative transfer equation in the schematic form. For more detailed study of the approximate methods we can recommend the reader the following papers on the same subject: Chandrasekhar, 1960; Sobolev, 1963; Malkevich, 1973; Ozisik, 1973; Marchuk, 1976; Marchuk *et al.*, 1986; Sabins, 1987; Thomas and Stamnes, 1999; Barichello *et al.*, 1998. The approximate methods are necessary from two points of view. First, they provide various simple methods for the solution of fairly complicated radiative transfer problems. In this case, however, their application is limited by the circumstance that the accuracy of the approximate method cannot be estimated without comparing it to the accurate solution or to the results obtained from accurate solutions of the Maxwell electromagnetic theory. Therefore, in using the approximate methods for studying particular natural media, some caution should be exercised, since the accuracy of any approximate method is not always clear enough. Second, in solving the reverse remote sensing problems, of principal significance is the possibility of describing the radiation of a studied natural medium by means of fairly simple analytical formulas. The use of numerical models (such as the Monte Carlo method) (Marchuk, 1976) does not allow to form the algorithms for reverse problems.

9.11.1 The spherical harmonics method

The spherical harmonics method enables one to obtain the approximate solution of the radiative transfer equation by using the initial assumption on a special form of an unknown solution. The physical basis for such a choice is the feature of electromagnetic energy scattering on particles, which allows for separating the angular and spatial variables in the Maxwell theory (Stratton, 1941). This method was first proposed by J. H. Jeans in 1917 in connection with the problem of radiative transfer in stellar atmospheres. The detailed description of the method of spherical harmonics as related to radiative transfer can be found in a series of papers (Chandrasekhar, 1960; Sobolev, 1963; Ozisik, 1973; Marchuk, 1976; Thomas and Stamnes, 1999; Barichello *et al.*, 1998).

Consider the radiative transfer equation for a planar layer of a grey medium under axial symmetry conditions:

$$\mu \frac{\partial I_\nu(\tau, \mu)}{\partial \tau} + I_\nu(\tau, \mu) = (1-\omega) I_{\mathrm{B}}[T(\tau)] + \frac{\omega}{2} \int_{-1}^{+1} p(\mu, \mu') I(\tau, \mu') \, \mathrm{d}\mu', \tag{9.96}$$

where it is supposed that the volume element scattering indicatrix can be presented in the form of expansion over the Legendre polynomials, but, unlike (9.46), with the other coefficients:

$$p(\mu, \mu') = \sum_{n=0}^{\infty} (2n+1) f_n P_n(\mu) P_n(\mu'). \tag{9.97}$$

Suppose that the unknown radiation intensity $I_\nu(\tau, \mu)$ can also be expanded in a series over the Legendre polynomials in the following special form, separating spatial and angular coordinates:

$$I(\tau, \mu) = \sum_{m=0}^{\infty} \frac{2m+1}{4\pi} P_m(\mu) \Psi(\tau). \tag{9.98}$$

If function $\Psi_m(\tau)$ is known, then the radiation intensity can be found from (9.98). For this reason we shall analyse in more detail the determination of function $\Psi_m(\tau)$. The substitution of expansions (9.97) and (9.98) into the basic equation (9.96) after some simplifications, determined by the orthogonality properties of the Legendre polynomials and by their recurrent formula (Ozisik, 1973), we obtain the system of ordinary differential equations with respect to function $\Psi_m(\tau)(m = 0, 1, 2, \ldots)$:

$$(m+1)\Psi'_{m+1} + m\Psi'_{m-1} + (2m+1)(1-\omega f_m)\Psi_m = 4\pi(1-\omega) I_{\mathrm{B}}[T(\tau)]\delta_{0m}, \tag{9.99}$$

where $f_0 = 1$ and prime denotes the differentiation with respect to τ.

For example, for the simplest isotropic scattering it is necessary to let in equation (9.96) all functions f_m equal to zero, except f_0, which is equal to unity.

Equations (9.96) form an infinite system of ordinary differential equations with an infinite number of unknown functions $\Psi_m(\tau)$. In practice, however, systems with a finite number of equations $m = N$ are considered, where the term $\Psi'_{m+1}(\tau)$ is neglected. The aforementioned procedure is certainly very important for the final solution and, hence, it should be substantiated from the physical point of view.

As a result, the following system of equations is obtained:

$$\begin{aligned} \Psi_1' + (1-\omega)\Psi_0 &= 4\pi(1-\omega)I_{\mathrm{B}}[T(\tau)] \\ 2\Psi_2' + \Psi_0' + 3(1-\omega f_1)\Psi_1 &= 0 \\ &\cdots \\ N\Psi_{N-1}' + (2N+1)(1-\omega f_N)\Psi_N &= 0 \end{aligned} \tag{9.100}$$

which represents the system of $N+1$ linear ordinary differential equations with $N+1$ unknown functions Ψ_0, $\Psi_1, \ldots, \Psi_N$ and is called the P_N approximation.

The solution of system (9.100), as known, can be written as a sum of the solution of the corresponding system of homogeneous equations and a particular solution. The latter, however, cannot be accurately determined until the function of black-body radiation intensity (i.e. the thermal regime inside the medium) is known. Let us find the solution of the system of homogeneous equations in the form of

$$\Psi_m^{\mathrm{H}}(\tau) = g_m \exp(k\tau); m = 0, 1, \ldots, N, \tag{9.101}$$

where g_m are arbitrary constants, and k are unknown exponent indices. The substitution of (9.101) into the system of homogeneous equations, obtained from (9.100), gives the following system of $N+1$ homogeneous algebraic equations with respect to coefficients g_m:

$$k[(m+1)g_{m+1} + mg_{m-1}] + (2m+1)(1-\omega f_m)g_m = 0, \tag{9.102}$$

where $m = 0, 1, 2, \ldots, N, f_0 = 1$ and $g_{N+1} = 0$.

In the case of isotropic scattering, we let $f_0 = 1$, and $f_m = 0 (m \neq 0)$.

Then (9.102) is simplified and takes the form of

$$k[(m+1)g_{m+1} + mg_{m-1}] + (2m+1)(1-\omega\delta_{0m})g_m = 0. \tag{9.103}$$

For the system of homogeneous algebraic equations (9.103) to possess a nontrivial solution, the determinant, composed of coefficients of the equations, should be equal to zero. Thus, as a result of performing the procedure mentioned, we obtain the allowable values of k_i for each value of ω. Then for each of k_i the set of $g_m(k_i)$ values $(m = 0, 1, 2, \ldots, N)$ is determined from equation (9.102), after which the solution of the system of homogeneous equations for isotropic scattering, obtained from (9.100), can be written in the form of

$$\Psi_m^{\mathrm{H}}(\tau) = \sum_{i=0}^{N} A_i g_m(k_i) \exp(k_i\tau); m = 0, 1, 2, \ldots, N. \tag{9.104}$$

The complete solution for function $Y_m(t)$ can be presented as

$$\Psi_m(\tau) = \Psi_m^{\mathrm{H}}(\tau) + \Psi^{\mathrm{P}}, \tag{9.105}$$

where the particular solution $\Psi^{\mathrm{P}}(\tau)$ depends on the spatial distribution of radiation intensity of an ideal black body, i.e. on the internal thermal regime in the medium. The unknown coefficients A appearing in (9.104) are found from the boundary conditions of a problem. Once functions $\Psi^{\mathrm{P}}(\tau)$ are determined, the unknown distribution of radiation intensity is found by formula (9.98). Here we note that there are also many other presentations of the solution of (9.104) related specifically to particular physical problems (Chandrasekhar, 1960; Sobolev, 1963; Ozisik, 1973; Barichello *et al.*, 1998).

As a particular case, we shall consider below the P_1 -approximation for isotropic scattering. This approximation is obtained from (9.100), if we accept $N = 1, f_m = \delta_{0m}$ and neglect the term $\mathrm{d}\Psi_2(\tau)/\mathrm{d}\tau$, i.e.

$$\left.\begin{aligned}\Psi_1'(\tau) + (1-\omega)\psi_0(\tau) = 4\pi(1-\omega)I_{\mathrm{B}}(T)\\ \Psi_0'(\tau) + 3\Psi_1(\tau) = 0.\end{aligned}\right\} \tag{9.106}$$

Rearranging the equations of system (9.106), we shall have the expressions for Ψ_0 and Ψ_1 separately:

$$\left.\begin{aligned}&\frac{\mathrm{d}^2\Psi_0}{\mathrm{d}\tau^2} = 3(1-\omega)[\Psi_0 - 4\pi I_{\mathrm{B}}(T)]\\ &\frac{\mathrm{d}^2\Psi_1}{\mathrm{d}\tau^2} = (1-\omega)\left[3\Psi_1 + 4\pi\frac{\mathrm{d}}{\mathrm{d}\tau}I_{\mathrm{B}}(T)\right].\end{aligned}\right\} \tag{9.107}$$

After determining function Ψ_0 from the solution of equation (9.107) and taking into account (9.106), we obtain the expression for the unknown intensity

$$T(\tau,\mu) = \frac{1}{4\pi}\left[\Psi_0(\tau) - \mu\frac{\mathrm{d}\Psi_0(\tau)}{\mathrm{d}\tau}\right] \tag{9.108}$$

The expressions for Ψ_0 will include both the boundary conditions and the thermal regime features. In the theory of stellar atmospheres (Sobolev, 1963) this approximation of a complete solution of the spherical harmonics method is called the Eddington approximation.

In remote sensing the spherical harmonics method (in the $P_2 - P_4$ approximation format) has been widely used in studying the thermal radiation of both small dispersed systems (non-precipitation clouds, aerosols) and medium dispersed systems (drizzle-type precipitation clouds), where the scattering still does not make a noticeable contribution to the total radiation balance of a system (see Chapter 10).

9.11.2 The Gaussian quadratures method

The Gaussian quadratures method makes it possible to obtain the approximate solution of the radiative transfer equation via the approximate presentation of integrals in the basic equation by the so-called Gaussian quadratures and subsequent transformation of the initial integro-differential equation into the system of ordinary differential equations.

We separate the unknown intensity into the direct (outgoing) component $I(\tau,\mu)$, $\mu \in (0,1)$, and the reverse component $I(\tau,\mu)$, $\mu \in (-1,0)$, as we have already done it in section 9.5, and we shall write the basic integro-differential equation in a slightly different (as compared to (9.50)–(9.53)) form:

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = (1-\omega) I_{\mathrm{B}}[T(\tau)] + \frac{\omega}{2}\left[\int_0^1 p(\mu,\mu') I(\tau,\mu')\,\mathrm{d}\mu' + \int_0^1 p(\mu,-\mu') I(\tau,-\mu')\,\mathrm{d}\mu'\right] \tag{9.109}$$

for $0 < \tau < \tau_0$, $\mu > 0$ and

$$\mu \frac{\partial I(\tau,-\mu)}{\partial \tau} + I(\tau,-\mu) = (1-\omega) I_{\mathrm{B}}[T(\tau)] + \frac{\omega}{2}\left[\int_0^1 p(\mu,-\mu') I(\tau,\mu')\,\mathrm{d}\mu' + \int_0^1 p(\mu,\mu') I(\tau,-\mu')\,\mathrm{d}\mu'\right] \tag{9.110}$$

for $0 < \tau < \tau_0$, $\mu > 0$.

Note that equations (9.109)–(9.110) are valid for positive values, i.e. $\mu \in (0,1)$, and two intensity components are distinguished by means of designations $I(\tau,\mu)$ and $I(\tau,-\mu)$.

The integral terms in above equations can be approximately presented by the sums with using the formula for the double Gaussian quadrature:

$$\int_0^1 f(\tau,\mu')\,\mathrm{d}\mu' \cong \sum_{j=1}^{N} a_j f(\tau,\mu_j);\ \mu > 0, \tag{9.111}$$

where a_j are weighting multipliers (Christoffel's coefficients), which are determined by the Gaussian quadrature formula, and μ_j are the discrete values of μ, which are determined by the Gaussian quadrature formula (Gradshteyn and Ryzhik, 2000).

The integro-differential equations (9.109) and (9.110) are transformed, by means of the N-point formula for the Gaussian quadrature, into the system of $2N$ ordinary differential equations with respect to intensities $I(\tau,\mu_i)$ and $I(\tau,-\mu_i)$ $(i = 1,2,\ldots,N)$. After some transformations equations (9.109) and (9.110) can be reduced to the form (Ozisik, 1973):

$$\frac{\mathrm{d}I(\tau,\mu_i)}{\mathrm{d}\tau} - \sum_{j=1}^{N} \alpha_{ij} I(\tau,\mu_j) - \sum_{j=1}^{N} \beta_{ij} I(\tau,-\mu_j) = \frac{1}{\mu_i}(1-\omega) I_{\mathrm{B}}[T(\tau)], \tag{9.112}$$

$$\frac{\mathrm{d}I(\tau,-\mu_i)}{\mathrm{d}\tau} + \sum_{j=1}^{N} \beta_{ij} I(\tau,\mu_j) + \sum_{j=1}^{N} \alpha_{ij} I(\tau,-\mu_j) = -\frac{1}{\mu_i}(1-\omega) I_{\mathrm{B}}[T(\tau)], \tag{9.113}$$

where $\mu_i \in (0,1)$, $i = 1, 2, \ldots, N$ and

$$\alpha_{ij} = \frac{1}{\mu_i}\frac{\omega}{2} a_j p(\mu_i, \mu_j) - \frac{\delta_{ij}}{\mu_j}, \tag{9.114}$$

$$\beta_{ij} = \frac{1}{\mu_i}\frac{\omega}{2} a_j p(\mu_i, -\mu_j). \tag{9.115}$$

Equations (9.112) and (9.113) represent the system of $2N$ linear ordinary differential equations with $2N$ unknown values $I(\tau, \mu_i)$ and $I(\tau, -\mu_i)$, $(i = 1, 2, \ldots, N)$, which should be solved simultaneously with $2N$ corresponding boundary conditions.

Suppose that the solution of a homogeneous system of equations, corresponding to the system of (9.112) and (9.113), can be written in the form of

$$I(\tau, \mu_i) = g_i(k) \exp(k\tau); I(\tau, -\mu_i) = g_i^*(k) \exp(-k\tau), \tag{9.116}$$

where $i = 1, 2, \ldots, N$.

After substitution of these solutions into homogeneous parts of equations (9.112) and (9.113) we shall obtain the system of $2N$ linear homogeneous algebraic equations with respect to $g_i(k)$ and $g_i^*(k)$ with k as a parameter. The allowable values of k are found from the condition, that the determinant composed of coefficients $g_i(k)$ and $g_i^*(k)$ becomes zero, if the resulting system of algebraic homogeneous equations has a nontrivial solution. Once the values of k_j are found, the algebraic equations are solved for each value of k_j $(j = 1, 2, \ldots, 2N)$ and the corresponding values of $g_i(k)$ and $g_i^*(k)(i = 1, 2, \ldots, N)$ are determined. The general solution of the system of equations (9.112) and (9.113) is written as a linear sum of general solutions of homogeneous equations and a particular solution:

$$I(\tau, \mu_i) = \sum_{j=1}^{N} c_j g_i(k_j) \exp(k_j \tau) + I^p, \tag{9.117}$$

$$I(\tau, -\mu_i) = \sum_{j=1}^{N} c_j g_i^*(k_j) \exp(k_j \tau) + I^p, \tag{9.118}$$

where c_j denotes $2N$ constants of integration, which should be found from $2N$ boundary conditions. For an isothermal medium the particular solution is found quite easily, and for a non-isothermal medium some special techniques can be used.

In microwave remote sensing the Gaussian quadratures method (true, in a slightly transformed form) has been applied in studying thermal radiation of almost all transparent media with inclusion of small dispersed scatterers (England, 1974, 1975). The internal scattering results in the effect of 'darkening' (or 'cooling') thermal radiation of the whole medium. In this case the 'cooling' effect will increase as the scattering albedo grows and medium's dielectric constant decreases. These effects should be expected from physical considerations (see section 9.9).

Natural media where these effects are possible include the glacial ice sheets of Antarctica and Greenland. As we have noted above (section 8.7), the negative frequency-selective spatial variation of thermal radiation of internal regions of Antarctica is caused by the effect of volume scattering of a glacial medium. Similar effects are quite possible for ground structures of the Moon and Mars as well.

9.11.3 Approximate formulas

In the cases where fairly simple algorithms for reverse microwave remote sensing problems can be formed (see Chapter 13), the necessity arises of using approximate formulas which will take into account the main scattering effects. However, the pure absorption approximation, which is often used in practice (see section 9.4), where the scattering effects are fully ignored, unsatisfactorily describes the radiation of a scattering layer (of the atmosphere) for optical thickness values greater than 1.5.

Earlier Basharinov *et al.* (1967) suggested a method for the description of the radiothermal emission of a planar isothermal layer by means of effective coefficients of transmission, q, and reflection, r, in the following simple form:

$$T_B = T_0(1 - q - r), \tag{9.119}$$

where T is the temperature of a medium,

$$q = \frac{(1 - r_0^2)\exp(-k\tau_0)}{1 - r_0^2 \exp(-2k\tau_0)}; r = r_0 \frac{1 - \exp(-2k\tau_0)}{1 - r_0 \exp(-2k\tau_0)}. \tag{9.120}$$

Here τ_0 is the total weakening in a scattering medium (for example, in a rain), and coefficients k and r_0 for a symmetrical scattering indicatrix are equal to

$$k = \sqrt{1 - \omega}; r_0 = \frac{1 - k}{1 - k}. \tag{9.121}$$

The expressions for q and r were obtained by Ambartsumyan (Sobolev, 1963) on the basis of one-dimensional model of the scattering of stellar atmospheres. In deriving (9.119) the Kirchhoff law was used, where it was supposed that, owing to scattering, the fraction of radiation equal to rT was reflected backwards and was as though 'extracted' from the basic energy balance. The comparison of results of a complete solution of the transfer equation and calculations by formula (9.119), carried out on paper by Basharinov *et al.* (1967), has shown that expression (9.119) gives overestimated values of radiobrightness temperatures, the distinctions not exceeding 15%.

Smirnov (1984) believes the following form of expression for the brightness temperature to be more correct:

$$T_B = T_0[1 - q - r(1 - q)]. \tag{9.122}$$

Here it was supposed that the part of radiation that was not dissipated in the medium possessed effective reflection. In this case the radiobrightness temperatures of outgoing T^+ and incident T^- radiation of the 'atmosphere – underlying surface' system can be written in the following form:

$$T^- = (1 - r)T_0(1 - q\exp(-\tau_0)) + r\kappa T_S, \tag{9.123}$$

$$T^+ = (1 - r)\lfloor \kappa T_S q \exp(-\tau_0) + T_0(1 - q\exp(-\tau_0)) + (1 - \kappa)T^- q\exp(-\tau_0)\rfloor, \tag{9.124}$$

where T_0 is the average temperature of a medium with scattering (the atmosphere with precipitation), τ_0 is the total absorption in a medium (in the atmosphere), κ is the underlying surface emissivity and T_S is the underlying surface temperature. The second term in (9.123) describes the part of the underlying surface radiation that is reflected from the precipitation layer owing to scattering.

The special modelling of radiation of a medium with scattering (the atmosphere with precipitation) by the Monte Carlo method (Smirnov, 1984) has demonstrated a good agreement between calculations by formulas (9.123) and (9.124) and modelling results. It was also shown in that paper, that the radiation intensity value only weakly depends on the form of a scattering indicatrix and can be approximately described by the single scattering model.

In conclusion, it should be mentioned that, depending on the physical and geometrical features of a specific problem, many of approximate formulas can be obtained for forming the algorithms of reverse problems.