AUTORESONANCE IN WEAKLY COUPLED OSCILLATORS

Agnessa Kovaleva

Space Research Institute, Russian Academy of Sciences
e-mail: agnessa_kovaleva@hotmail.com

Abstract. We investigate autoresonance (AR) in the system of two coupled oscillators. Two classes of autoresonant problems are investigated analytically and numerically: (1) a periodic force with constant frequency is applied to the Duffing oscillator with slowly time-decreasing linear stiffness; (2) the time-independent nonlinear oscillator is excited by a force with slowly increasing frequency. In both cases, stiffness of the linear oscillator and coupling remains constant, and the system is initially engaged in resonance. It is demonstrated that in the system of the first type AR occurs in both oscillators but in the system of the second type AR in the nonlinear oscillator occurs along with irregular small oscillations of the linear oscillator

1. Introduction

The phenomenon of the permanent growth of energy in a classical anharmonic oscillator subjected to periodic forcing with slow variations of forcing and/or resonant frequencies was first used in applications to particle acceleration [1, 2]. The term “autoresonance” (AR), introduced [3] in the context of cyclotron resonance stability, seems to be more convenient in the study of the increasing amplitudes of the resonant process.

Theoretical approaches, experimental evidence and applications of AR in different fields of natural science, from plasmas to planetary dynamics, have been reported in numerous works (see, e.g., [4 - 6], and references therein). Recent advances in the study of AR in two or three degree-of-freedom (2D and 3D) are excitations of continuously phase-locked plasma waves with laser beams [4, 7 - 10], particle transport in a weak external field with slowly changing frequency [11, 12], etc.

The behavior of each oscillator in the multi-dimensional array can principally differ from the dynamics of a single oscillator. Below we illustrate this effect by an example of the 2D system.

2. AR in the system of the first type

In this section the model of two coupled oscillators is considered:

$$m_1 \frac{d^2u_1}{dt^2} + c_1u_1 + c_{12}(u_1 - u) = 0,$$
$$m_0 \frac{d^2u}{dt^2} + C(t)u + ku^3 + c_{12}(u - u_1) = A\cos\omega t, \quad (1)$$

where u and u_1 correspond to the absolute displacements of the nonlinear and linear oscillators, respectively; the parameter $C(t) = c_0 - (k_1 + k_2t)$, $k_{1,2} > 0$. The system is assumed to be initially at rest.

We define the small parameter ε as $c_{12}/c_1 = 2\varepsilon << 1$. Next, assuming weak nonlinearity and taking into account resonance properties of the system, we redefine the parameters as follows:

$$c_1/m_1 = c_0/m_0 = \omega^2,\ \tau_0 = \omega t,\ \tau_1 = \varepsilon\tau_0,\ A = \varepsilon m\omega^2F,$$
$$k_1/c_0 = 2\varepsilon s,\ k_2/c_0 = 2\varepsilon^2 b\omega,\ k/c_0 = 8\varepsilon\alpha,\ c_{12}/c_r = 2\varepsilon\lambda_r,\ r = 0, 1. \quad (2)$$

In these notations, the equations of motion are rewritten as:

$$\frac{d^2u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1(u_1 - u) = 0,$$
$$\frac{d^2u_0}{d\tau_0^2} + (1 - 2\varepsilon\zeta(\tau_1))u + 2\varepsilon\lambda_0(u - u_1) + 8\varepsilon\alpha u^3 = 2\varepsilon F\sin\tau_0, \quad (3)$$

where $\zeta(\tau_1) = s + b\tau_1$. The multiple scales decomposition [13] is applied to construct explicit asymptotic solutions of Eqs. (3). To this end, we introduce the complex amplitudes

$$Y_r = (v_r + iu_t)e^{-i\tau_0}, r = 0,1, \quad (4)$$

where Y and Y_1 are sought in the form of expansions

$$Y_r = \varphi_r(\tau_1) + \varepsilon\varphi_r^{(1)}(\tau_0, \tau_1) + \ldots \quad (5)$$

To keep the notations simpler, we employ the transformations

$$\tau = s\tau_1,\ \psi_r(\tau_1) = \varphi_r(\tau_1)/\Lambda,\ \Lambda = (s/3\alpha)^{1/2},$$
$$f = F/s\Lambda,\ \mu_r = \lambda_r/s,\ \beta = b/s^2,\ \zeta_0(\tau) = 1 + \beta\tau, \quad (6)$$

Inserting (4) - (6) into (3), after some little algebra we obtain the dimensionless equations for the complex slow amplitudes ψ_0 and ψ_1

$$\frac{d\psi_1}{d\tau} - i\mu_1(\psi_1 - \psi_0) = 0, \psi_1(0) = 0,$$
$$\frac{d\psi_0}{d\tau} - i\mu_0(\psi_0 - \psi_1) + i(\zeta_0(\tau) - |\psi_0|^2)\psi_0 = -if, \psi_0(0) = 0. \quad (7)$$

The real-valued amplitudes and phases of oscillations are defined as $a_r = |\psi_r|$, $\Delta_r = \arg\psi_r$, $r = 0, 1$. The thorough derivation of the leading-order equations is presented in [14-16].

The following parameters are used for numerical simulations:

$$\varepsilon = 0.05,\ \beta = 0.05,\ \mu_0 = 0.02,\ \mu_1 = 0.25;\ f = 0.34. \tag{8}$$

It was shown [17] that the system with chosen parameters admits AR. Figure 1 depicts the amplitudes of both oscillators.

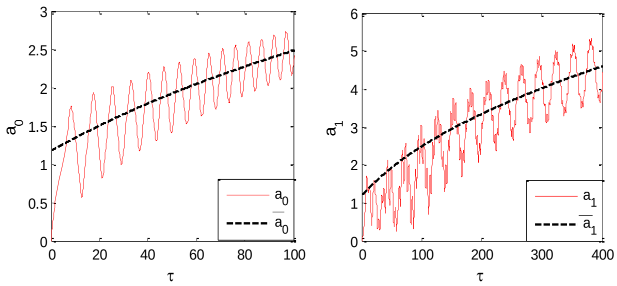

Fig. 1. Amplitudes of oscillations in the systems of the 1st type

It is seen from Fig.1 that the solutions of system (7) can be presented in the form

$$\psi_r(\tau) = \overline{\psi}_r(\tau) + \tilde{\psi}_r(\tau), \tag{9}$$

where $\tilde{\psi}_r(\tau)$ represents small fast fluctuations near the respective quasi-steady state $\overline{\psi}_r$, $r = 0, 1$.

The states $\overline{\psi}_r$ can be computed from (7) at "frozen" ζ_0 under the conditions $d\overline{\psi}_r / d\tau = 0$, $r = 0, 1$. Under these assumptions we obtain that $\overline{\psi}_0 = \overline{\psi}_1$, and

$$(\zeta_0 - |\overline{\psi}_0|^2)\overline{\psi}_0 = -f\ . \tag{10}$$

Hence $\mathrm{Im}\,\overline{\psi}_0 = 0$, $\mathrm{Re}\,\overline{\psi}_0 = |\overline{\psi}_0|$; $|\overline{\psi}_r| = \overline{a}_r, \overline{a}_1 = \overline{a}_0$. It follows from (2.13) that $\overline{a}_0^2 \approx \zeta_0 \to \beta\tau$ if $f << 2\zeta_0$. Note that this asymptotic limit was earlier obtained for a single oscillator [17].

3. AR in the system of the second type

In this section we briefly analyse AR in the system with constant parameters driven by an external force with the slowly changing frequency. The equations of motion are reduced the form

$$\frac{d^2u_1}{d\tau_0^2}+u_1+2\varepsilon\lambda_1(u_1-u_0)=0,$$
$$\frac{d^2u_0}{d\tau_0^2}+u_0+2\varepsilon\lambda_0(u_0-u_1)+8\varepsilon\alpha u_0^3=2\varepsilon F\sin(\tau_0+\theta(\tau_1)), \qquad (11)$$
$$\frac{d\theta}{d\tau_1}=\zeta(\tau_1).$$

with initial conditions u_r = 0, v_r = 0 at τ_0 = 0, r = 0, 1. Reproducing transformations (4) - (6) and introducing the variables $\psi_r=\phi_r e^{i\theta_0(\tau)}$ we obtain the following equations for the slow complex amplitudes ϕ_r:

$$\frac{d\varphi_1}{d\tau}+i\zeta_0(\tau)\varphi_1-i\mu_1(\varphi_1-\varphi_0)=0, \varphi_1(0)=0,$$
$$\frac{d\varphi_0}{d\tau}-\iota\mu_0(\varphi-\varphi_1)+i(\zeta_0(\tau)-|\varphi_0|^2)\varphi_0=-if, \varphi_0(0)=0 \qquad (12)$$

with coefficients (6). The real-valued amplitudes a_r and phases Δ_r are defined as $a_r = |\phi_r|$, $\Delta_r = \arg\phi_r$, r =0, 1.

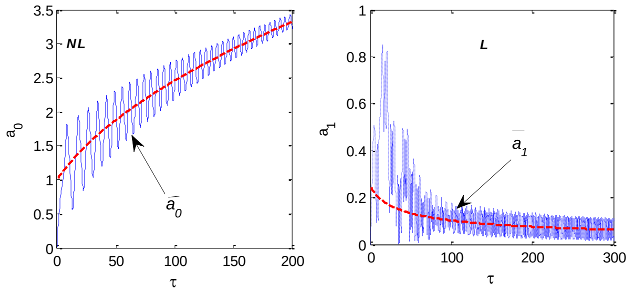

Fig. 2. Amplitudes of oscillations in the systems of the 2nd type

Figure 2 depicts numerical results for the system with parameters (8) It is seen that the amplitude of the nonlinear oscillator is very close to its analogue presented in Fig.1 but the behavior of the linear oscillator drastically differs from regular AR in Fig. 1.

As shown above, the quasi-steady state $\bar{\phi}_0$ is sought as a solution of system (12) at “frozen” ζ_0 and $d\phi_0/d\tau = 0$. It is easy to prove that the function $\bar{\phi}_0$ satisfies the equation analogous to (10), and $\bar{a}_0=|\bar{\phi}_0|$.

If the solution ϕ_0 is presented in the form $\phi_0 = \bar{\phi}_0 + \tilde{\phi}_0$, then the equation for ϕ_1 can be rewritten as

$$\frac{d\phi_1}{d\tau} + i[\zeta_0(\tau) - \mu_1]\phi_1 = -i\mu_1(\bar{\phi}_0 + \tilde{\phi}_0), \phi_1(0) = 0 . \qquad (13)$$

Hence $\phi_1 = \bar{\phi}_1 + \tilde{\phi}_1$, where $\bar{\phi}_1 = -[\mu_1/(\zeta_0 - \mu_1)]\bar{\phi}_0$. Since $\bar{\phi}_0 \approx \sqrt{\zeta_0}$, the slowly varying function $\bar{\phi}_1$ and the corresponding quasi-steady amplitude are given by:

$$\bar{\phi}_1(\tau) \approx -\mu_1/\sqrt{\zeta_0(\tau)} \to 0, \; \bar{a}_1(\tau) = |\bar{\phi}_1(\tau)| \approx \mu_1/\sqrt{\zeta_0(\tau)} \to 0$$

as $\tau \to \infty$. Dashed lines in Fig. 2 depict the functions $\bar{a}_0 = \sqrt{\zeta_0}$ and $\bar{a}_1(\tau) = \mu_1/\sqrt{\zeta_0(\tau)}$.

A key conclusion from these results is that the passage through resonance in the system of this type does not provide oscillations with growing energy in the attached oscillator; furthermore, the amplitude of linear oscillations decreases when the frequency detuning increases.

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