A fully relativistic twisted accretion disc around a Kerr black hole

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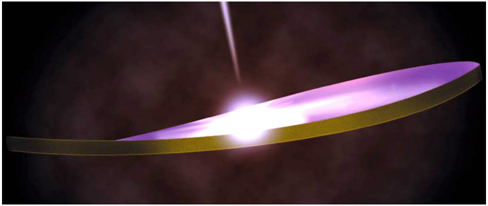
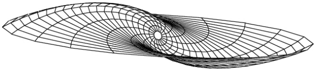


Overview of previous analytical studies



- Bardeen & Petterson (1975) — a first quantitative study, the disc is considered as a collection of "rigid" rings, the equations obtained are however quantitatively wrong, say, they do not conserve angular momentum.
- Petterson (1977,1978)
  Hatchett, Begelman & Sarazin (1981) — twisted coordinate systems were considered for Newtonian discs, the equations are improved but not quite...
- Papaloizou & Pringle (1983) — a first correct consideration. It was shown that one has to consider velocity/density perturbations in a self-consistent picture. It was stressed that these perturbations determine evolution/shapes of stationary configurations for nearly Keplerian, low viscosity discs.

Overview of analytical previous studies

- Papaloizou & Lin (1995) — bending waves, the case of low viscosity.
- Ivanov & Illarionov (1997) — stationary solution, low viscosity and post-Newtonian corrections, it was found that a low viscosity stationary disc exhibits radial oscillations instead of alignement with the equatorial plane in case of disk gas rotating in the same sense as the black hole.
- Demianski & Ivanov (1997) — a full system of equations with post-Newtonian corrections describing a low viscosity twisted disc.
- Lubow, Ogilvie & Pringle (2002) — time dependant evolution of such discs.

Overview of analytical previous studies

The previous studies showed that a nearly Keplerian twisted disc has two different types of time evolution and stationary configurations for $\alpha > h/r$ and $\alpha < h/r$, where α is the well known Shakura-Sunyaev parameter.

h/r is assumed to be small, it is of the order of $10^{-2} - 10^{-3}$.

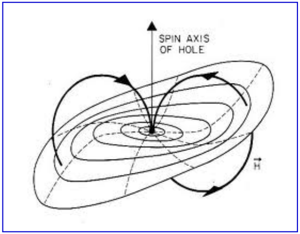
Overview of analytical previous studies

The case of $\alpha > h/r$

In this case a stationary twisted disc always aligns with the equatorial plane of a rotating black hole at radii smaller than or of the order of

$$R_1 \sim \alpha^{2/3} a^{2/3} (r/h)^{4/3} R_g ,$$

where R_g is the gravitational radius, a is the rotational parameter of the black hole.



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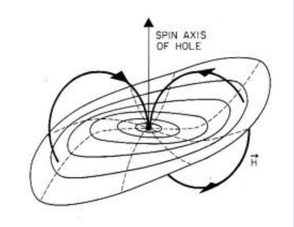


The time evolution has a character of diffusion, with characteristic time scale

$$t_1 = \alpha (r/h)^2 \Omega^{-1},$$

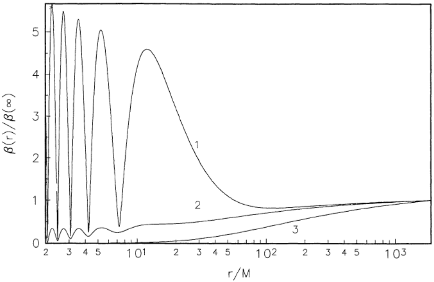
where the Keplerian angular frequency $\Omega = \sqrt{GM/r^3}$, M is the mass of the black hole.

Overview of previous analytical studies

The case of $\alpha < h/r$

When $a > 0$ a stationary twisted disc experiences radial oscillations of its inclination angle β (The angle between a unit vector normal to the disc's rings and the equatorial plane of the black hole). When $a < 0$ it aligns with the equatorial plane. The scale of oscillation/alignement

$$R_2 \sim a^{2/5}(r/h)^{4/5}R_g$$



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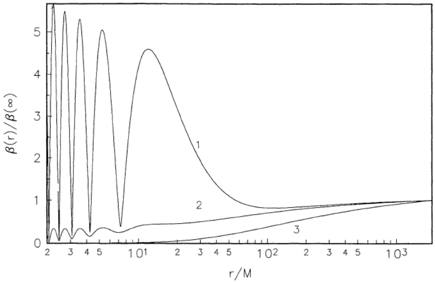


The time evolution is wave-like with a characteristic time scale of order of the 'sound' time scale

$$t_2 \sim (r/h)\Omega^{-1}$$

Metric

The Kerr metric of a slowly rotating black hole in the linear approximation in rotational parameter a $a \ll 1$:

$$ds^2 = (1 - 2M/R)dt^2 - (1 - 2M/R)^{-1}dR^2 - R^2(d\theta^2 + sin^2\theta d\phi^2) + 4a\frac{M}{R}sin^2\theta \, d\phi \, dt$$

Let us change to the "isotropic" radial coordinate:

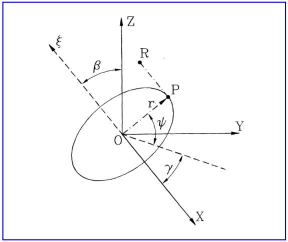
$$R = R_I \left(1 + \frac{M}{2R_I}\right)^2$$

and get a metric with the spacial line element proportional to the Cartesian spatial line element:

$$ds^2 = K_1^2 dt^2 + 2ar^2 K_1 K_3 d\phi dt - K_2^2(dr^2 + dz^2 + r^2 d\phi^2)$$

Twisted coordinates

$$\begin{pmatrix} \tau \\ r\cos\psi \\ r\sin\psi \\ \xi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma & 0 \\ 0 & -\sin\gamma & \cos\gamma & \beta \\ 0 & \beta\sin\gamma & -\beta\cos\gamma & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$



1. $|\beta(\tau, r)| \ll 1$, $\gamma(\tau, r)$ are functions to be determined
2. $\Psi_1 = \beta\cos\gamma$, $\Psi_2 = \beta\sin\gamma \implies \mathbf{W} = \beta e^{i\gamma}$
3. change to $\varphi = \psi + \gamma$
4. $Z = \beta\sin\psi = \Psi_1\sin\varphi - \Psi_2\cos\varphi$, $\quad U = \dot{Z}$, $\quad W = Z'$

Equations of motion

Relativistic hydrodynamics

The law of mass conservation: $(\rho U^i)_{;i} = 0$

The law of energy-momentum conservation: $T^{ik}_{;k} = 0$

Both laws of consevation are splitted on unperturbed and perturbed parts. It may be shown that in the linear approximation in angle β equations describing the unperturbed part can be reduced to the ones of the Novikov-Thorne model.

$$T^{ik} = (\epsilon + p)U^i U^k - pg^{ik} + T^{ik}_{\nu} - U^i q^k - U^k q^i,$$

$$g^{ik} = diag\{1, -1, -1, -1\}, \qquad \epsilon = \rho + \epsilon_{th}$$

The viscous part is $T^{ik}_{\nu} = 2\eta\sigma^{ik}$

The shear tensor is $\sigma^{ik} = \frac{1}{2}(U^i_{;j}P^{jk} + U^k_{;j}P^{ji}) - \frac{1}{3}U^j_{;j}P^{ik}$

The projection tensor is $P^{ik} = g^{ik} - U^i U^k$

q^i is the energy flux, η is the dynamical viscosity

A reduced system of equations

- Consider an isothermal density vertical distribution

$$\rho = \rho_c \exp(-\frac{\xi^2}{2h^2})$$

- The velocity perturbations have now the form

$$v^\varphi = \xi(A_1 \sin\varphi + A_2 \cos\varphi) \quad v^r = \xi(B_1 \sin\varphi + B_2 \cos\varphi)$$

- Let us introduce complex notation

$$\mathbf{A} = A_2 + iA_1, \quad \mathbf{B} = B_2 + iB_1 \quad \text{and} \quad \mathbf{W} = \beta e^{i\gamma}$$

A reduced system of equations

The "horizontal" part of twist equations can be rewritten as

$$\dot{\mathbf{A}} - (i-\alpha)\Omega\mathbf{A} + \frac{\kappa^2}{2\tilde{\Omega}}\mathbf{B} = -\frac{3}{2} i\alpha K_1 (U^\tau)^2 U^\varphi \Omega \mathbf{W}'$$

$$\dot{\mathbf{B}} - (i-\alpha)\Omega\mathbf{B} - 2\tilde{\Omega}\mathbf{A} = -(i+\alpha) U^\varphi \Omega\, \mathbf{W}'$$

Characteristic frequencies of a slightly perturbed free circular motion in the Schwarzschild metrics are

$$\Omega = R^{-3/2}, \quad \kappa^2 = R^{-3}\left(1 - \frac{6}{R}\right), \quad \tilde{\Omega} = \frac{R-3}{R^2 (R-2)^{1/2}}$$

A reduced system of equations

The "vertical" part of twist equations can be rewritten as

$$\dot{\mathbf{W}} - i\Omega_{LT}\mathbf{W} + \frac{3}{2}\alpha\delta^2 \frac{K_1^2}{K_2} U^\varphi \left(U^\tau - K_1 (rK_2)^{1/2} \frac{U^\varphi}{D} \right) \mathbf{W}' =$$
$$\frac{\delta^2 K_1^3 U^\varphi}{2r^{1/2} K_2^{3/2} D} \frac{\partial}{\partial r} \left\{ r^{3/2} K_2^{1/2} \frac{D}{K_1^2 U^\tau U^\varphi} \left((i+\alpha)\mathbf{B} + \alpha U^\varphi \mathbf{W}' \right) \right\}$$

which contains an additional characteristic frequency of the problem, namely, the Lense-Thirring frequency

$$\Omega_{LT} = \Omega - \Omega_\perp = 2aR^{-3}$$

where $\Omega_\perp$ is the frequency of a free vertical harmonic oscillations in the equatorial plane of the Kerr black hole.

$$\delta(r) = h(R)/R = \delta_* K_1^{3/5} K_2^{1/20} (U^\tau)^{-4/5} D^{1/5} r^{1/20}$$

and

$$D = 1 - \frac{\sqrt{6}}{\sqrt{R}} - \frac{\sqrt{3}}{2\sqrt{R}} \ln \frac{(\sqrt{R}-\sqrt{3})(3+2\sqrt{2})}{(\sqrt{R}+\sqrt{3})}.$$

Note that $D(R = R_{ms}) = 0$.

An equation for stationary shapes

Setting all time derivatives in the twist equations to zero we get

$$\frac{K_1}{R^{1/2} D} \frac{d}{dR} \left(\frac{R^{3/2} D}{K_1 U^\tau} f^*(\alpha, R) \frac{d\mathbf{W}}{dR} \right) - 3\alpha U^\tau (1 - D^{-1}) \frac{d\mathbf{W}}{dR} + \frac{4ia}{\delta^2 K_1^3 R^3 U^\varphi} \mathbf{W} = 0$$

where $*$ stands for the complex conjugate and

$$f(\alpha, R) = (1 + \alpha^2 - 3i\alpha K_1^2) \, \frac{R(i-\alpha)}{\alpha R(\alpha + 2i) - 6} + \alpha$$

The solutions have two independent parameters

$$\alpha \quad \rightarrow [0, 1]$$

$$\tilde{\delta} = \delta_* / \sqrt{|a|} \quad \rightarrow (0, \infty)$$

Analytical solution: an almost inviscid case

Let us set formally $\alpha = 0$, thus obtaining

$$\frac{d}{dR}\left(b\frac{d}{dR}\mathbf{W}\right) + \lambda\mathbf{W} = 0,$$

where $b \propto D$, $\lambda \propto D/\delta^2$

Close to R_{ms} $\Longrightarrow$ $x = R - R_{ms}$ and $D \simeq x^2/72$

The twisted disc near $R_{ms} = 3R_g$ is described by

$$\frac{1}{x^2}\frac{d}{dx}x^2\frac{d}{dx}\mathbf{W} + \chi x^{-4/5}\mathbf{W} = 0 \qquad (*)$$

The regular solution of $(*)$ gives a boundary condition at R_{ms}

$$\mathbf{W} = Cx^{-1/2}J_{5/6}(5/3\sqrt{\chi}x^{3/5})$$

The analytical solution: an almost inviscid case

The analytical solution for the shape of twisted accretion disc for $a > 0$ and $\tilde{\delta} \ll 1$

Close to R_{ms}

$$\mathbf{W} = C_1 x^{-1/2} J_{5/6}(5/3\sqrt{\chi}x^{3/5})$$

WKBJ-oscillations in the relativistic region

$$\mathbf{W} \approx \frac{C_2}{(\lambda b)^{1/4}} \cos\left(\int_{R_{ms}}^{R} \sqrt{\lambda/b} dR + \phi_{WKBJ}\right)$$

For $R \gg R_{ms}$

$$\mathbf{W} = C_3 x_1^{3/2}(A_1 J_{-3/5}(C_4 R^{-5/4}) + A_2 J_{3/5}(C_4 R^{-5/4}))$$

Resonant solutions: a self-warping disc or a tilting doll

The relation between inclination angles at R_{ms} and far from the black hole

$$\mathbf{W}_\infty = C_{tot}(\tilde{\delta})\mathbf{W}_0$$

Average behaviour

$$C_{tot} \sim \tilde{\delta}^{43/30}$$

$C_{tot} \rightarrow 0$ at discrete values of

$$\tilde{\delta}_k \simeq \frac{\pi}{2}(73/30 + 2k)\,,$$

$k = 0, 1, 2, ...$

Numerical results

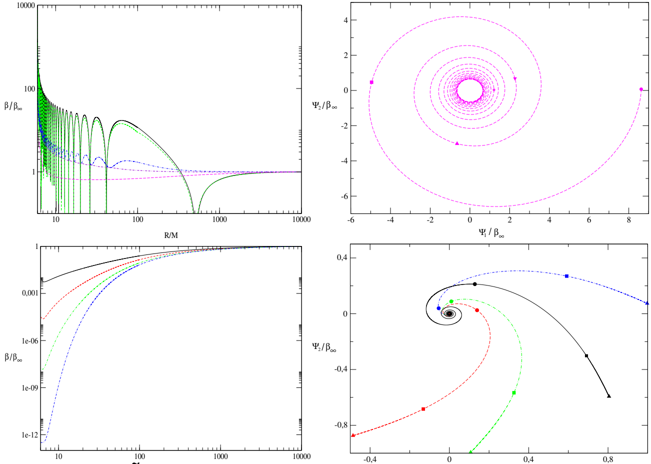


Figure: Upper panel: $\tilde{\delta} = 10^{-2}$, $\alpha = 0, 10^{-4}, 10^{-3}, 10^{-2}$. The right plot $\beta(R)$, the left plot - projection of the unit vector perpendicular to the disc rings onto the equatorial plane Lower panel: the same as the upper one but for larger $\alpha = 0.05$, 0.1, 0.2 and 1.

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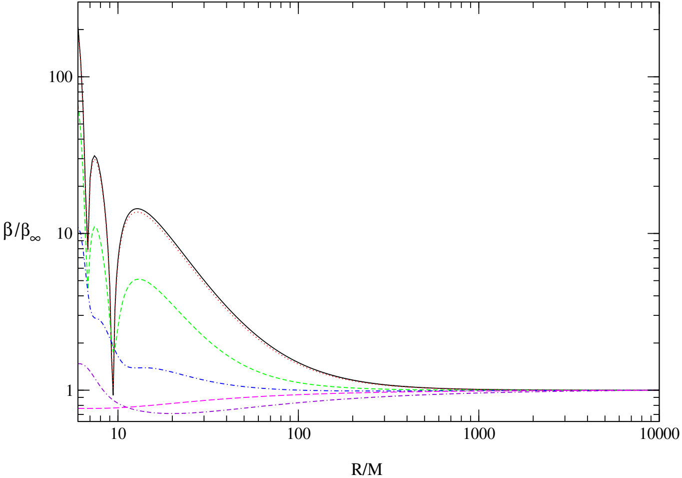

Figure: $\beta(R)$ for $\tilde{\delta} = 0.1$ and $\alpha = 0$, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1} and 1.

Numerical results

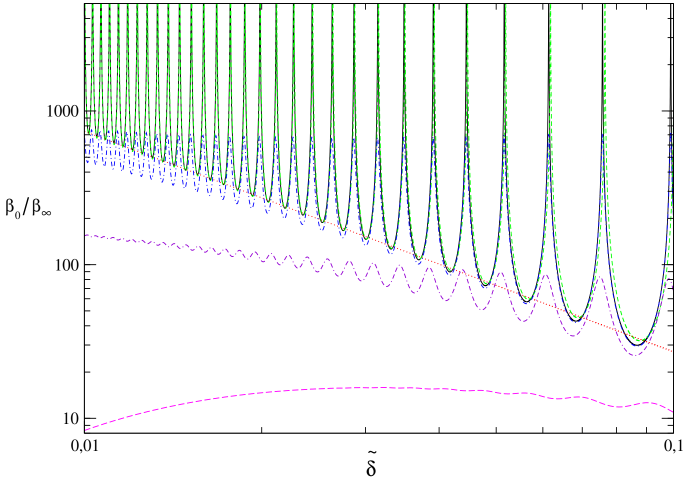

Figure: $\beta(R = R_{ms})$ as a function of $\tilde{\delta}$. $\beta(R = \infty) = 1$, $\alpha = 0,\ 10^{-4},\ 10^{-3},\ 10^{-2}$.

Numerical results

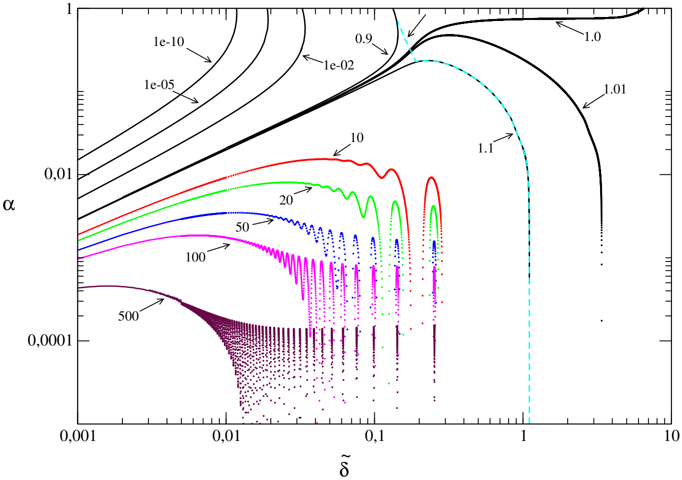

Figure: Levels of constant $\beta(R = R_{ms})$ on the plane $(\tilde{\delta}, \alpha)$, $\beta(R = \infty) = 1$.

Summary

- Dynamical equations describing the evolution and stationary configurations of a fully relativistic thin twisted disc have been derived assuming $\beta \ll 1$ and $a \ll 1$
- For the simple Novikov-Thorne model of a flat disc with a constant value of the Shakura-Sunyaev parameter α equations can be further simplified. The final twist equations are formulated for three complex variables **W**, **A** & **B** determining the geometry of the disc and shear velocities induced by warp. Under certain assumption **A** can be expressed through **B**, and the disc dynamics can be described by a pair of equations for **W** and **B**.
- The stationary configurations can be fully described by two parameters — α & $\tilde{\delta} = \delta_* / \sqrt{a}$.
- An analytical theory of stationary disc has been constructed for the case $a > 0$ & $\tilde{\delta} \ll 1$. The disc exhibit prominent oscillations of the inclination angle with R which can grow indefinitely while $\tilde{\delta} \to 0$. Also there are specific "resonant" solutions for discrete values for $\tilde{\delta}$.
- For a moderate value of the viscosity parameter α the Bardeen-Petterson effect is absent. The disc remains to be twisted in the vicinity of a black hole. The disc can align with the equatorial plane of a black hole only in the case of a large value of α and a sufficiently small $\tilde{\delta}$.

Remarks

Possible further developments.

- Time-dependant solutions of twist equations — temporal activity of accretion discs in relativistic regime. Quasi-normal modes with $\omega << \Omega$.
- One may include the next order terms in h/r in equations. This must be related to twisted slim discs. The problem of peculiarity near the last stable orbit R_{ms} should be considered.
- Generalisation of stationary solutions up to $a \sim 1$.
- Feedback effects of warp and twist: self-irradiation of the disc, vertical structure modified by shear velocities etc.
- Calculation of light curves, spectral features and other observational manifestations for particular astrophysical sources.