Динамическая стабилизация несферических тел относительно неограниченного коллапса

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Динамическая устойчивость сферических звезд

Dynamic stability of spherical stars is determined by an average adiabatic power $\gamma = \frac{\partial \log P}{\partial \log \rho}|_S$. For a star with a density distribution $\rho = \rho_0 \varphi(m/M)$, the star in the newtonian gravity is stable against dynamical collapse when $\int_0^R (\gamma - \frac{4}{3}) P \frac{dm}{\varphi(m/M)} > 0$ (Zeldovich and Novikov, 1967; Bisnovatyi-Kogan, 1989). This approximate criterium becomes exact for adiabatic stars with constant γ. Here ρ_0 is a central density, M is a stellar mass, m is the mass inside a Lagrangian radius r, so that $m = 4\pi \int_0^r \rho r^2 dr$, $M = m(R)$, R is a stellar radius. Collapse of a spherical star may be stopped only at stiffening of the equation of state, like neutron star formation at late stages of evolution, or formation of fully ionized stellar core with $\gamma = \frac{5}{3}$ at collapse of clouds during star formation. Without such stiffening a spherical star in the newtonian theory would collapse into a point with infinite density (singularity).

Невращающийся однородный трехосный эллипсоид

Let us consider 3-axis ellipsoid with semi-axes $a \neq b \neq c$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{1}$$

and uniform density ρ. A mass m of the uniform ellipsoid is written as (V is the volume of the ellipsoid)

$$m = \rho\, V = \frac{4\pi}{3}\,\rho\, abc\,. \tag{2}$$

Let us assume a linear dependence of velocities on coordinates

$$v_x = \frac{\dot{a}x}{a}\,, \quad v_y = \frac{\dot{b}y}{b}\,, \quad v_z = \frac{\dot{c}z}{c}\,. \tag{3}$$

The gravitational energy of the uniform ellipsoid is defined as:

$$U_g = -\frac{3Gm^2}{10}\int\limits_0^\infty \frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}. \tag{4}$$

The equation of state $P = K\rho^\gamma$ is considered here, with $\gamma = 4/3$.

A spherical star with $\gamma = 4/3$ collapses to singularity at small enough K, and we show here, how deviations from a spherical form prevent formation of any singularity.

For $\gamma = 4/3$, the thermal energy $E_{th} \sim V^{-1/3} \sim (abc)^{-1/3}$, and the value

$$\varepsilon = E_{th}(abc)^{1/3} = 3\left(\frac{3m}{4\pi}\right)^{1/3} K$$

remains constant in time.

A Lagrange function of the ellipsoid is written as

$$L = U_{kin} - U_{pot}\,, \quad U_{pot} = U_g + E_{th}\,, \tag{5}$$

$$U_{kin} = \frac{1}{2}\rho\int\limits_V (v_x^2 + v_y^2 + v_z^2)\,dV = \frac{m}{10}\,(\dot{a}^2 + \dot{b}^2 + \dot{c}^2)\,, \tag{6}$$

$$E_{th} = \frac{\varepsilon}{(abc)^{1/3}}\,. \tag{7}$$

Уравнения движения

$$\ddot{a} = -\frac{3Gm}{2}\, a \int\limits_0^\infty \frac{du}{(a^2+u)\Delta} + \frac{5}{3m}\,\frac{1}{a}\,\frac{\varepsilon}{(abc)^{1/3}}\,,$$

$$\ddot{b} = -\frac{3Gm}{2}\, b \int\limits_0^\infty \frac{du}{(b^2+u)\Delta} + \frac{5}{3m}\,\frac{1}{b}\,\frac{\varepsilon}{(abc)^{1/3}}\,,$$

$$\ddot{c} = -\frac{3Gm}{2}\, c \int\limits_0^\infty \frac{du}{(c^2+u)\Delta} + \frac{5}{3m}\,\frac{1}{c}\,\frac{\varepsilon}{(abc)^{1/3}}\,,$$

$$\Delta^2 = (a^2+u)(b^2+u)(c^2+u)\,.$$

Безразмерные уравнения движения

$$\ddot{a} = \frac{3}{2a^2(1-k^2)}\left[k - \frac{\arccos k}{\sqrt{1-k^2}}\right] + \frac{1}{a}\frac{\varepsilon}{(a^2c)^{1/3}}, \qquad (24)$$

$$\ddot{c} = -\frac{3}{a^2(1-k^2)}\left[1 - \frac{k\arccos k}{\sqrt{1-k^2}}\right] + \frac{1}{c}\frac{\varepsilon}{(a^2c)^{1/3}} \qquad (25)$$

for the oblate spheroid $k = c/a < 1$,

$$\ddot{a} = -\frac{3}{2a^2(k^2-1)}\left[k - \frac{\cosh^{-1} k}{\sqrt{k^2-1}}\right] + \frac{1}{a}\frac{\varepsilon}{(a^2c)^{1/3}}, \qquad (26)$$

$$\ddot{c} = \frac{3}{a^2(k^2-1)}\left[1 - \frac{k\cosh^{-1} k}{\sqrt{k^2-1}}\right] + \frac{1}{c}\frac{\varepsilon}{(a^2c)^{1/3}} \qquad (27)$$

for the prolate spheroid $k = c/a > 1$, and

$$\ddot{a} = -\frac{1-\varepsilon}{a^2} \qquad (28)$$

for the sphere, where the equilibrium corresponds to $\varepsilon_{eq} = 1$. Near the spherical shape we should use expansions around $k = 1$, what leads to equations of motion valid for both oblate and prolate cases

$$\ddot{a} = -\frac{1-\varepsilon}{a^2} + \left(\frac{\varepsilon}{3} + \frac{3}{5}\right)\frac{1-k}{a^2},$$

$$\ddot{c} = -\frac{1-\varepsilon}{a^2} + \left(\frac{4\varepsilon}{3} - \frac{4}{5}\right)\frac{1-k}{a^2}. \qquad (29)$$

Безразмерная полная энергия (безразмерный гамильтониан)

$$H = \frac{\dot{a}^2}{5} + \frac{\dot{c}^2}{10} - \frac{3}{5a}\frac{\arccos k}{\sqrt{1-k^2}} + \frac{3}{5}\frac{\varepsilon}{(a^2c)^{1/3}}, \quad \text{(oblate)}$$

$$H = \frac{\dot{a}^2}{5} + \frac{\dot{c}^2}{10} - \frac{3}{5a}\frac{\cosh^{-1} k}{\sqrt{k^2-1}} + \frac{3}{5}\frac{\varepsilon}{(a^2c)^{1/3}}, \quad \text{(prolate)}$$

$$H = \frac{3\dot{a}^2}{10} - \frac{3}{5a}(1-\varepsilon), \quad \text{(sphere)} \qquad (31)$$

$$H = \frac{\dot{a}^2}{5} + \frac{\dot{c}^2}{10} - \frac{3}{5a}\left(1 + \frac{\delta}{3} + \frac{2\delta^2}{15}\right) + \frac{3\varepsilon}{5a}\left(1 + \frac{\delta}{3} + \frac{2\delta^2}{9}\right),$$

$\delta = 1 - k$, (around the sphere), $|\delta| \ll 1$.

Результаты численных расчетов

Sphere:
$\varepsilon = 1$ – полная энергия равна нулю ($H = 0$), радиус произволен
$\varepsilon < 1$ – сфера коллапсирует в сингулярность
$\varepsilon > 1$ – разрушение звезды с разлетом на бесконечность

Spheroid:
$\varepsilon = 0$ – слабая сингулярность, образование блина
$\varepsilon > 0$ – **коллапса в сингулярность не происходит**:

При $\varepsilon \geq 1$ полная энергия $H > 0$ – происходит разлет на бесконечность
При $\varepsilon < 1$ полная энергия $H < 0$ – **устанавливается колебательный режим**, и сфероид динамически стабилизируется относительно коллапса в сингулярность. Причем в зависимости от начальных условий колебания могут быть как регулярными периодическими, так и хаотическими.

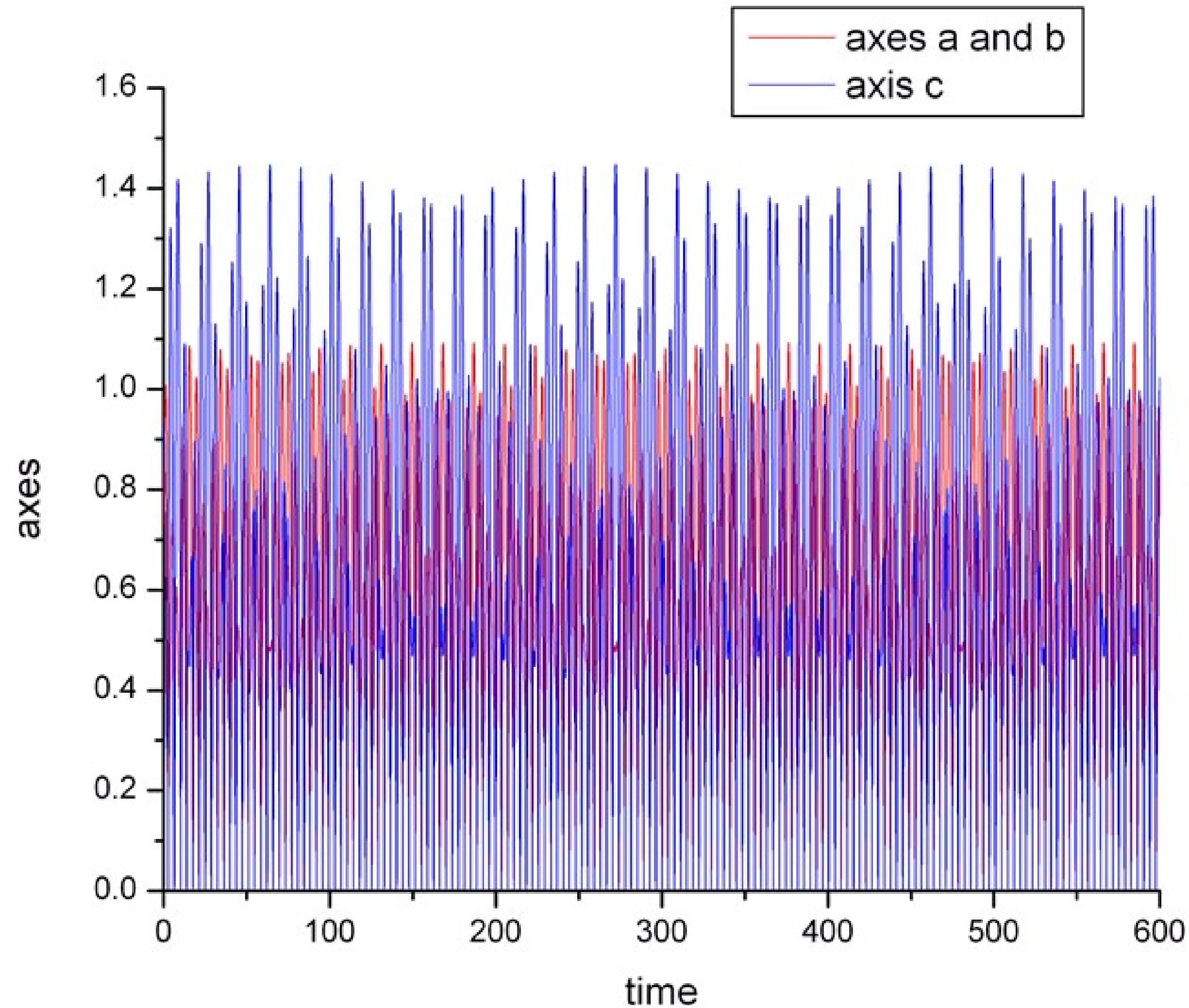


Figure 1. Example of regular motion of spheroid with $H = -1/5$, $\varepsilon = 2/3$. This motion corresponds to full line on the Poincaré map in Fig.4.

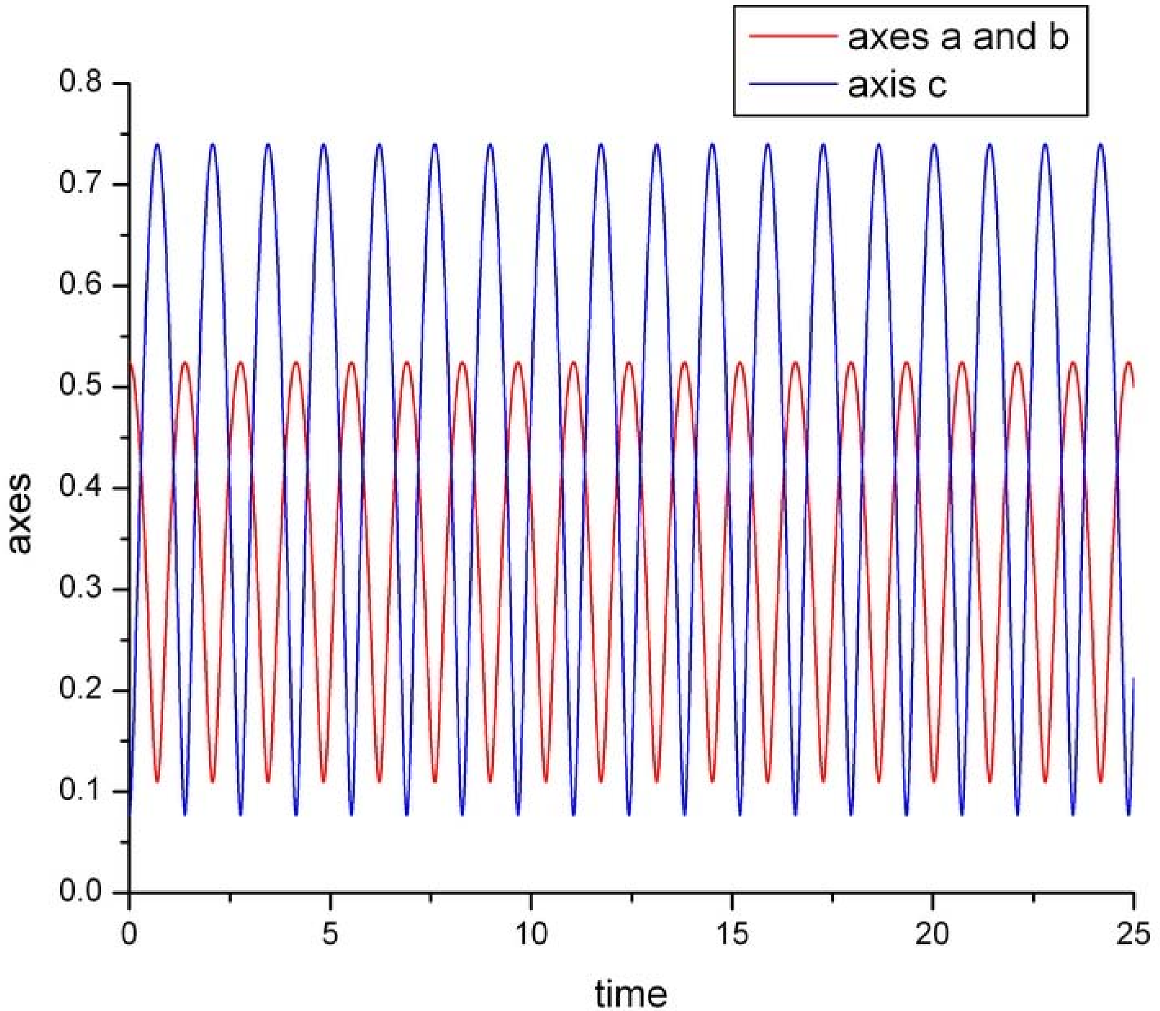


Figure 2. Example of regular motion of spheroid with $H = -1/5$, $\varepsilon = 2/3$. This motion corresponds to the point inside the regular region on the Poincaré map in Fig.4.

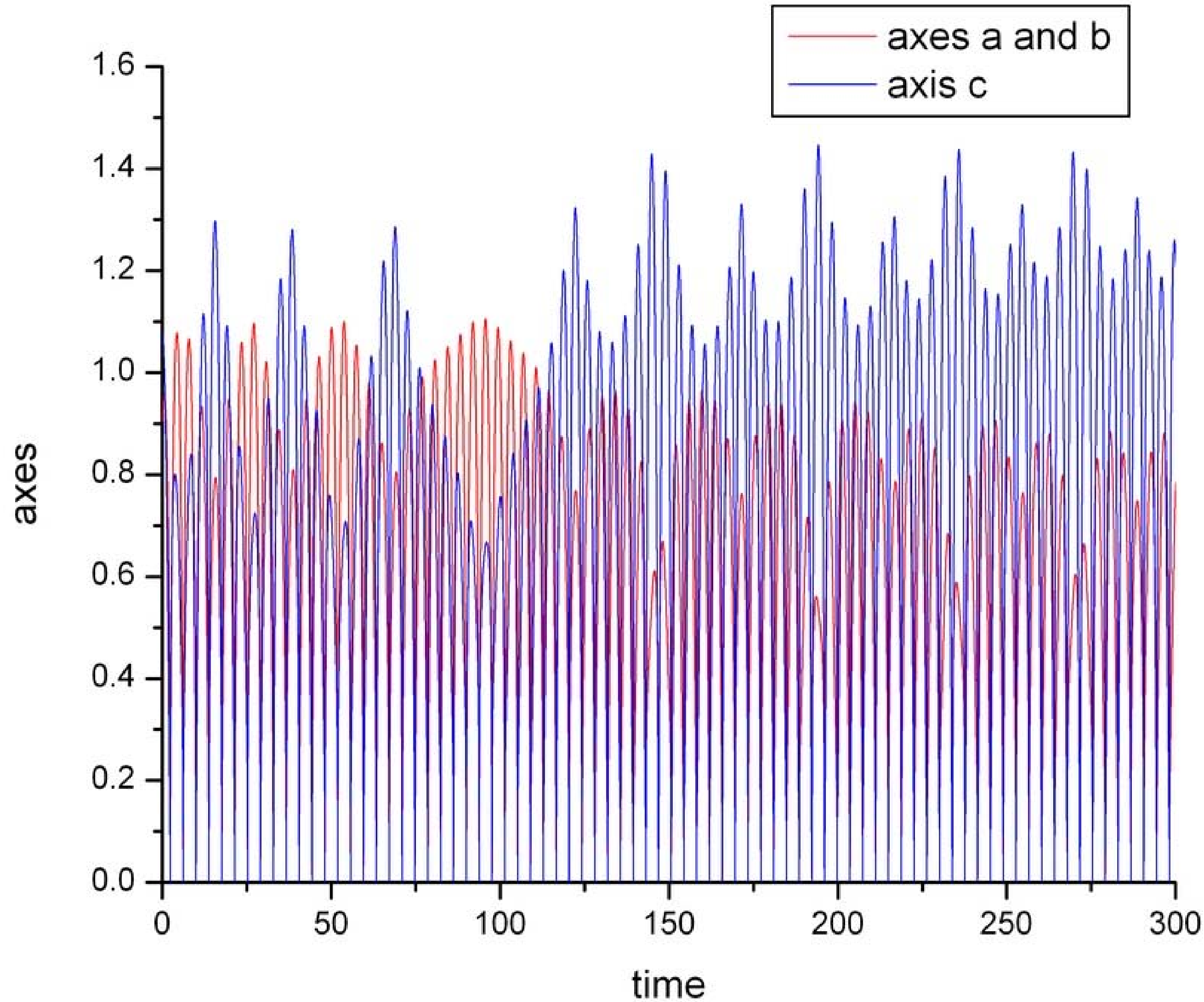


Figure 3. Example of chaotic motion of spheroid with $H = -1/5$, $\varepsilon = 2/3$. This motion corresponds to gray points on the Poincaré map in Fig.4.

4 The Poincaré section

To investigate regular and chaotic dynamics we use the method of Poincaré section and obtain the Poincaré map for different values of the total energy H. Let us consider a spheroid with semi-axes $a = b \neq c$. This system has two degrees of freedom. Therefore in this case the phase space is four-dimensional: a, $\dot{a}$, c, $\dot{c}$. If we choose a value of the Hamiltonian H_0, we fix a three-dimensional energy surface $H(a, \dot{a}, c, \dot{c}) = H_0$. During the integration of the equations (18)-(21) which preserve the constant H, we fix moments t_i, when $\dot{c} = 0$. At these moments there are only two independent values (i.g. a and $\dot{a}$), because the value of c is determined uniquely from the relation for the hamiltonian at constant H. At each moment t_i we put a dot on the plane $(a, \dot{a})$

For the same values of H and ε we solve equations of motion (18)-(21) at initial $\dot{c} = 0$, and different a, $\dot{a}$. For each integration we put the points on the plane $(a, \dot{a})$ at the moments t_i. These points are the intersection points of the trajectories on the three-dimensional energy surface with a two-dimensional plane $\dot{c} = 0$, called the Poincaré section.

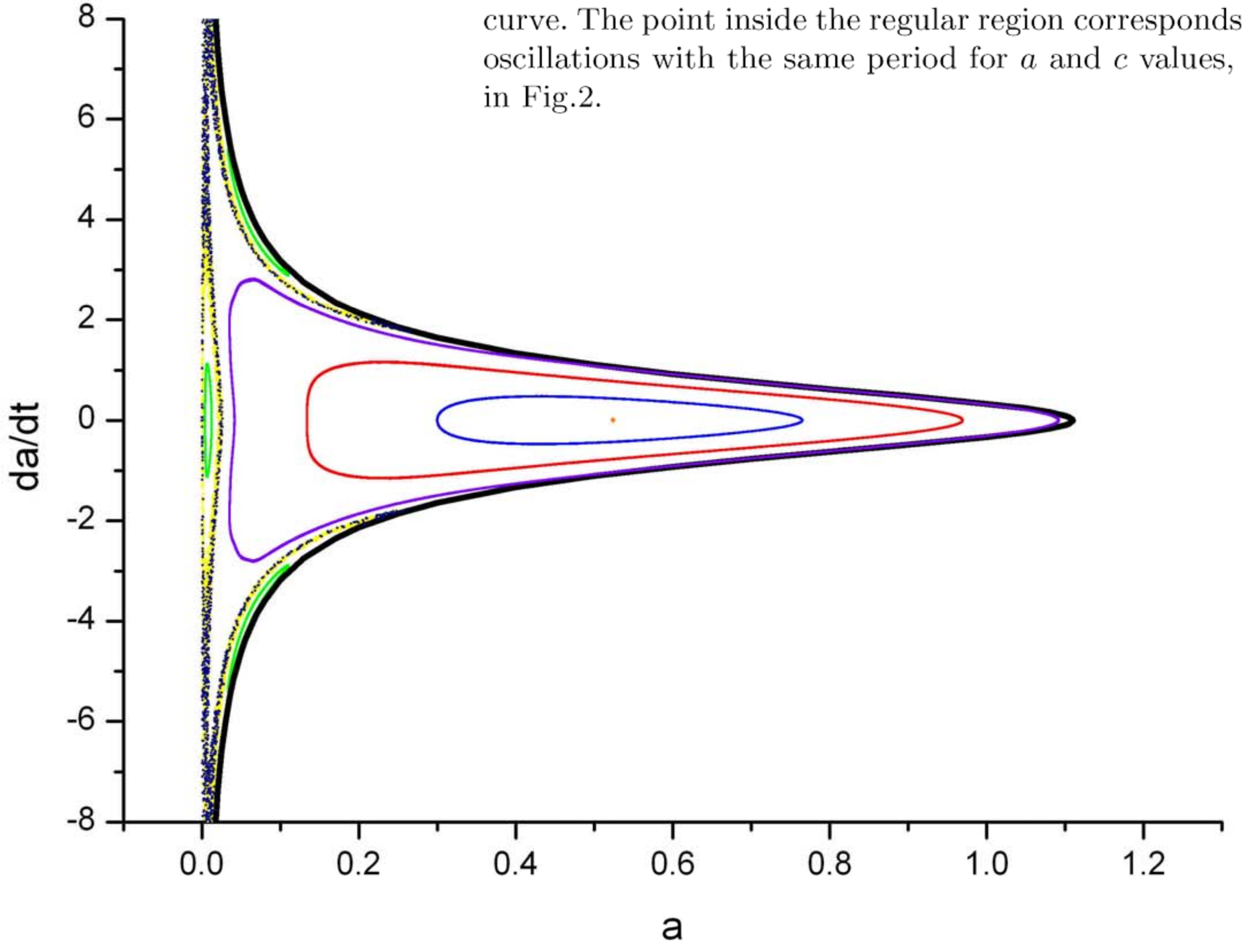


Figure 4. The Poincaré map for five regular and two chaotic trajectories in case of $H = -1/5$, $\varepsilon = 2/3$. The $(a, \dot{a})$ values are taken in the minimum of c. Full black line is the bounding curve. The point inside the regular region corresponds to coherent oscillations with the same period for a and c values, represented in Fig.2.

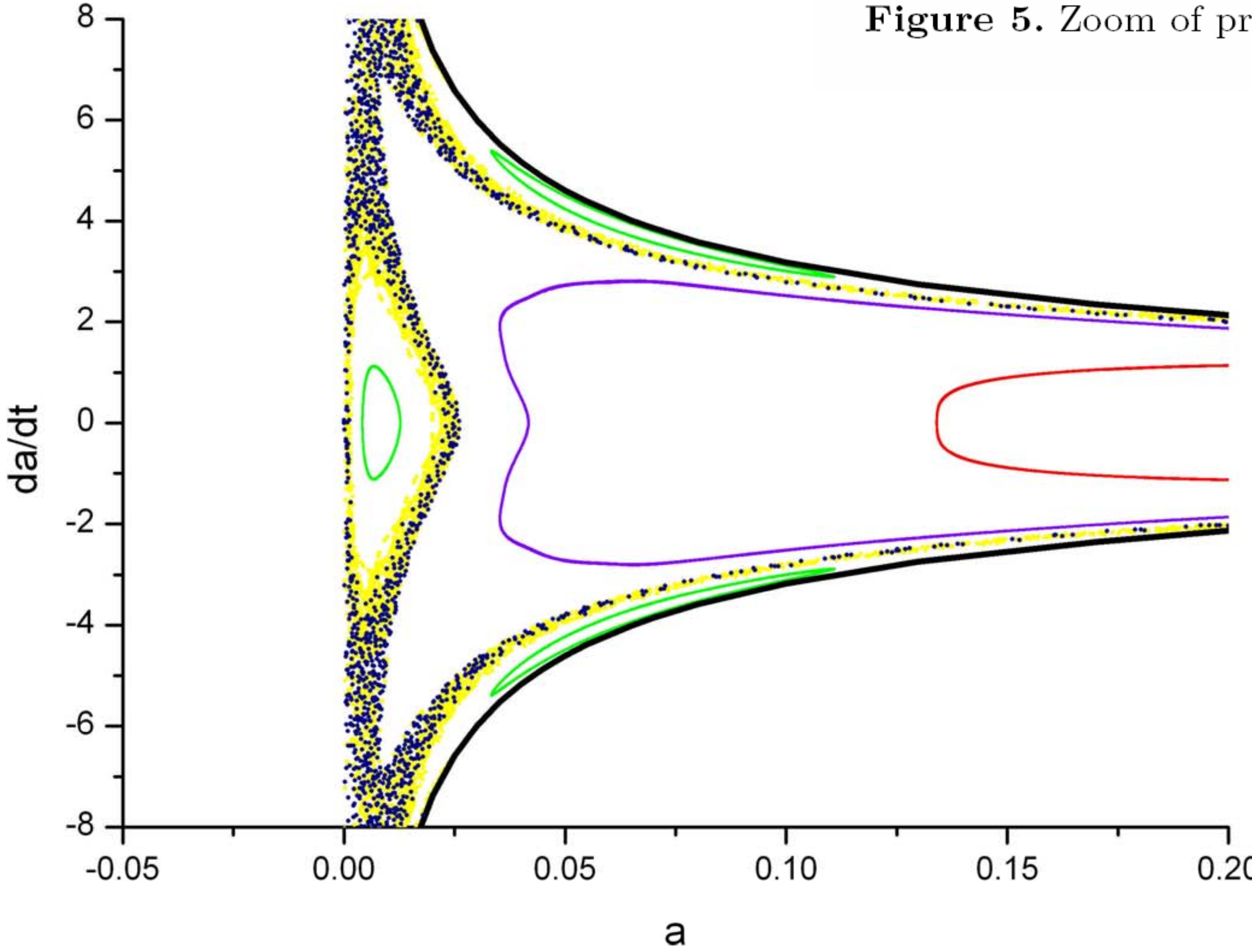


Figure 5. Zoom of previous figure.

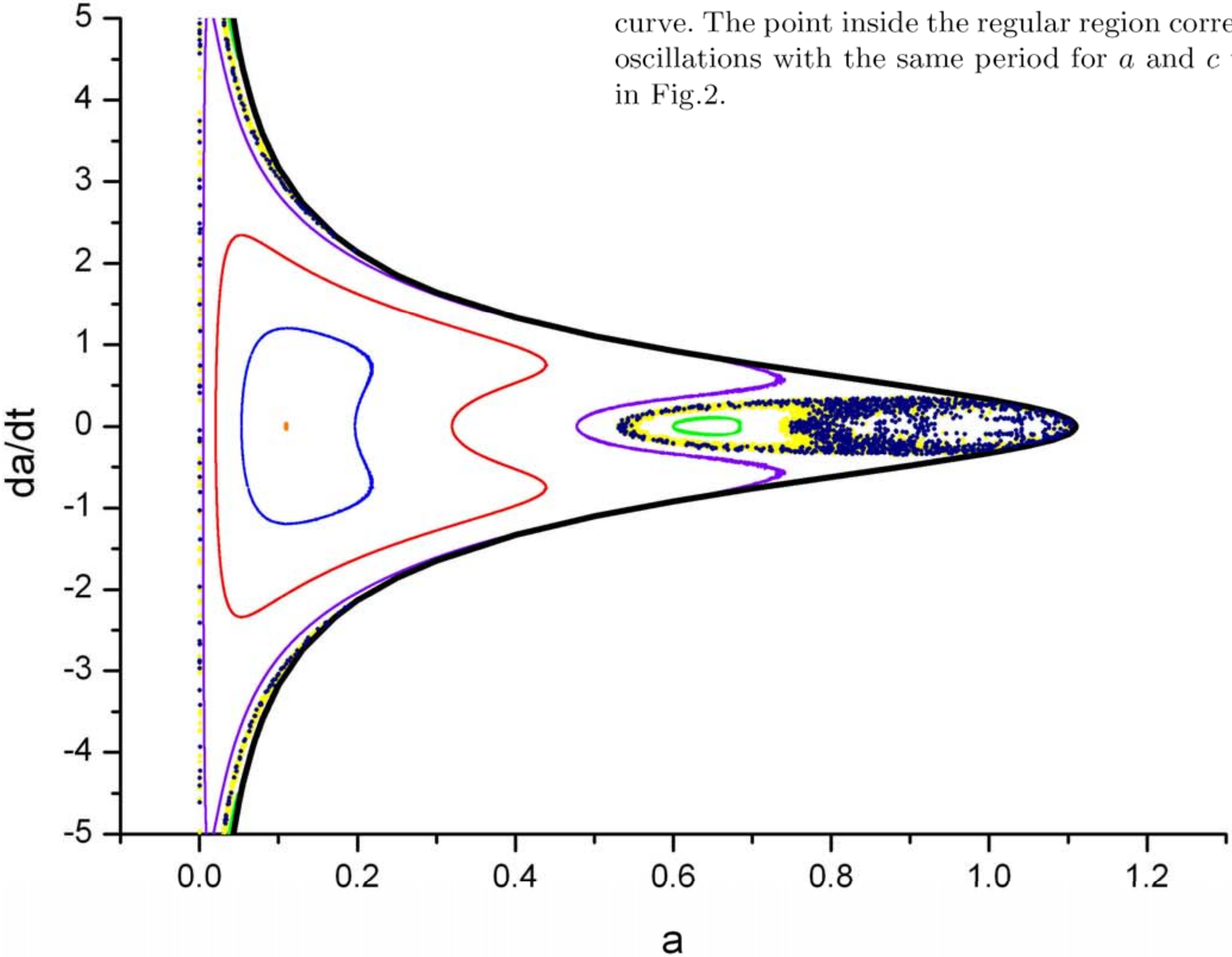


Figure 6. The Poincaré map for five regular and two chaotic trajectories in case of $H = -1/5$, $\varepsilon = 2/3$. The $(a, \dot{a})$ values are taken in the maximum of c. Full black line is the bounding curve. The point inside the regular region corresponds to coherent oscillations with the same period for a and c values, represented in Fig.2.

Хаотическое движение

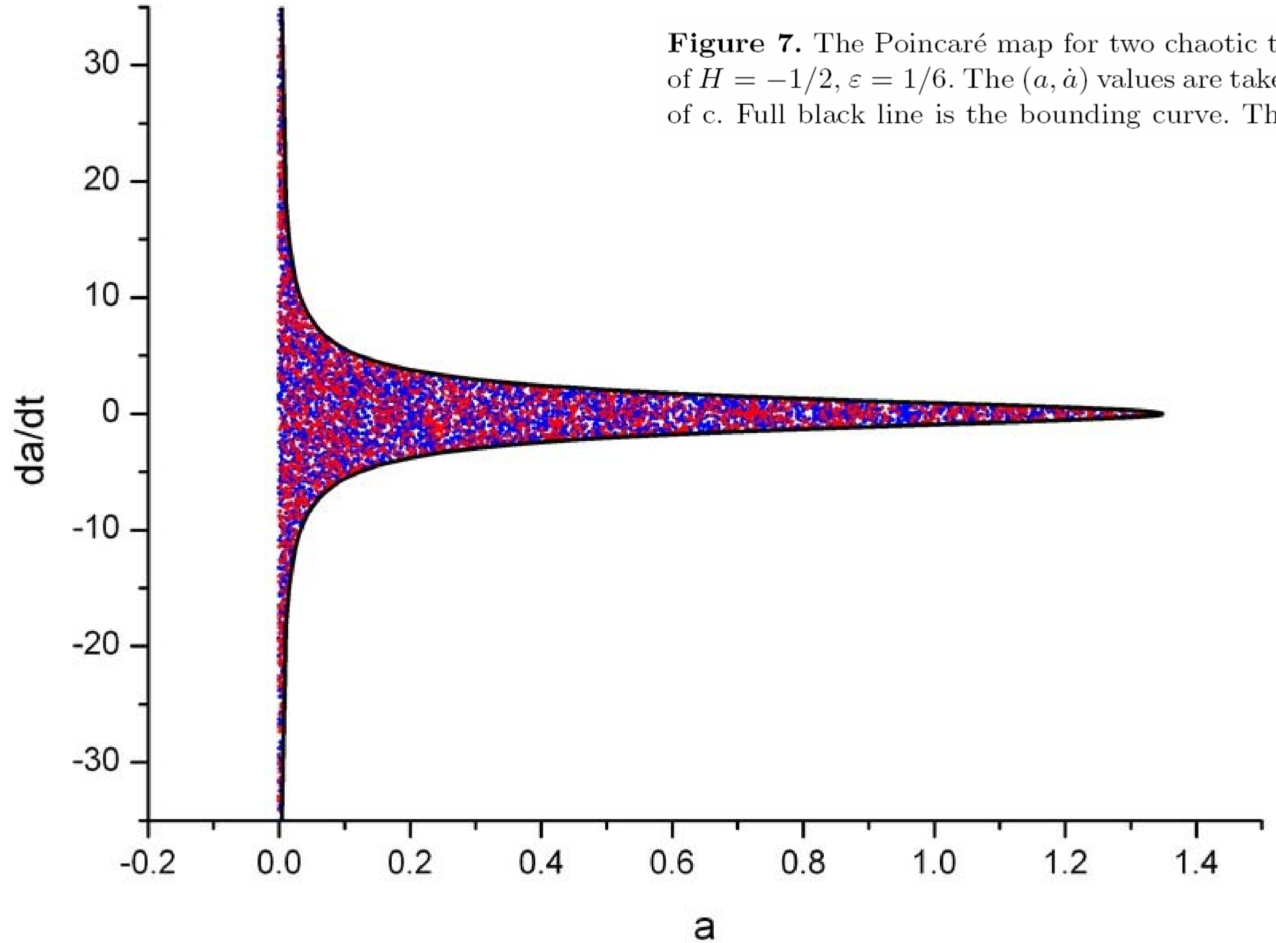


Figure 7. The Poincaré map for two chaotic trajectories in case of $H = -1/2$, $\varepsilon = 1/6$. The $(a, \dot{a})$ values are taken in the minimum of c. Full black line is the bounding curve. The point inside the

Регулярное движение

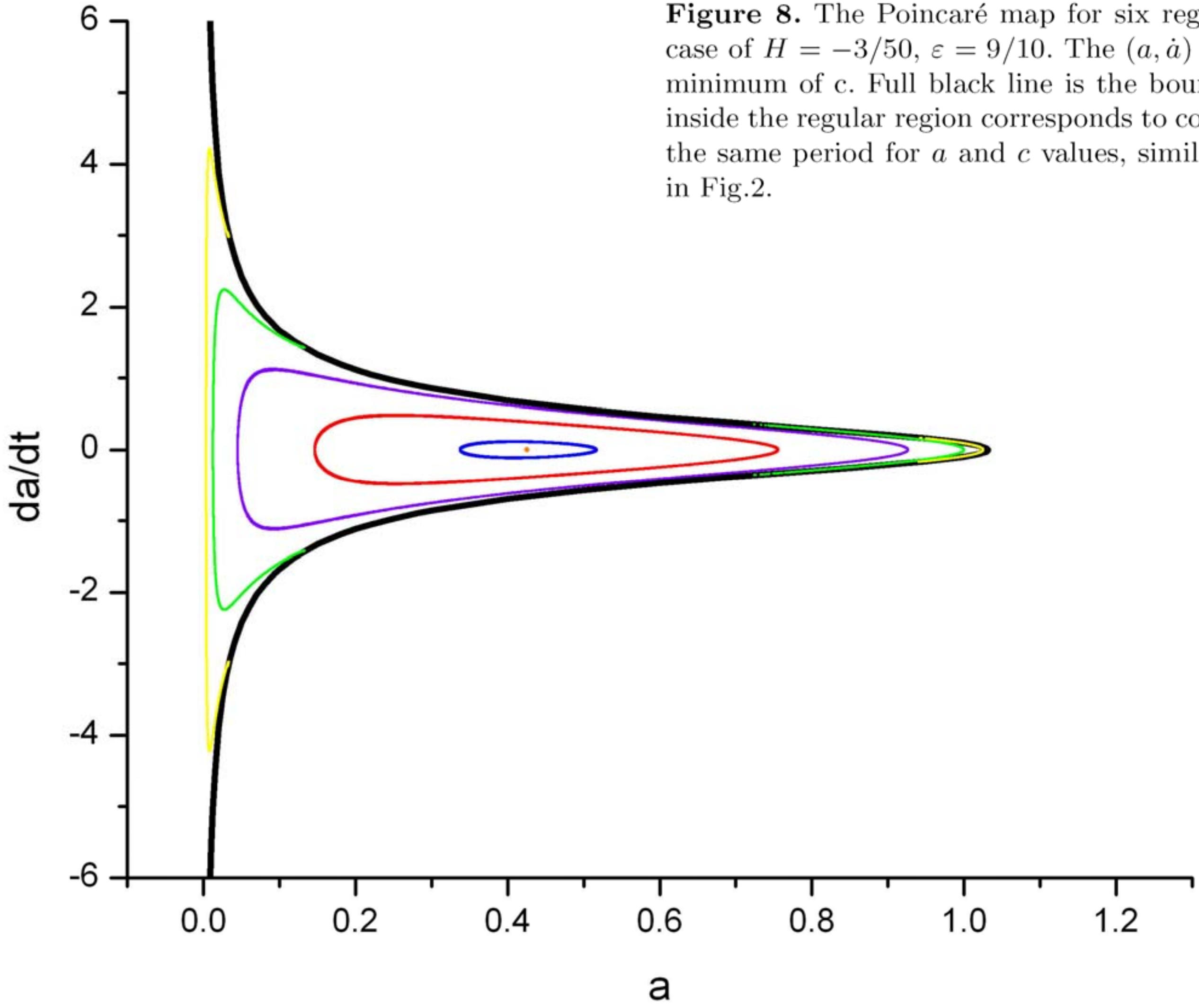


Figure 8. The Poincaré map for six regular trajectories in the case of $H = -3/50$, $\varepsilon = 9/10$. The $(a, \dot{a})$ values are taken in the minimum of c. Full black line is the bounding curve. The point inside the regular region corresponds to coherent oscillations with the same period for a and c values, similar to those represented in Fig.2.

Случай гамма=6/5

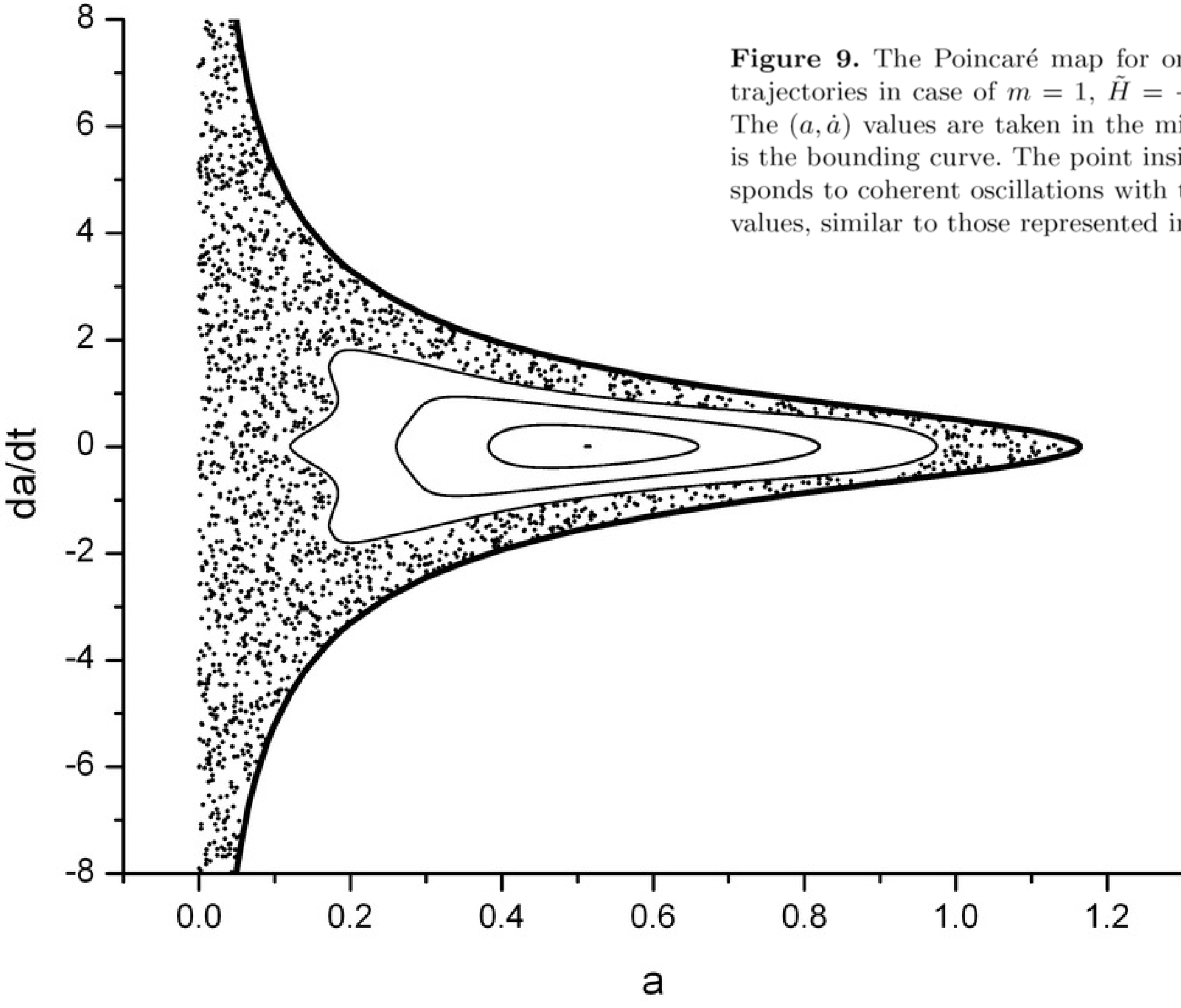


Figure 9. The Poincaré map for one chaotic and four regular trajectories in case of $m = 1$, $\tilde{H} = -3/50$, $\tilde{\varepsilon} = 27/50$, see (37). The $(a, \dot{a})$ values are taken in the minimum of c. Full black line is the bounding curve. The point inside the regular region corresponds to coherent oscillations with the same period for a and c values, similar to those represented in Fig.2.

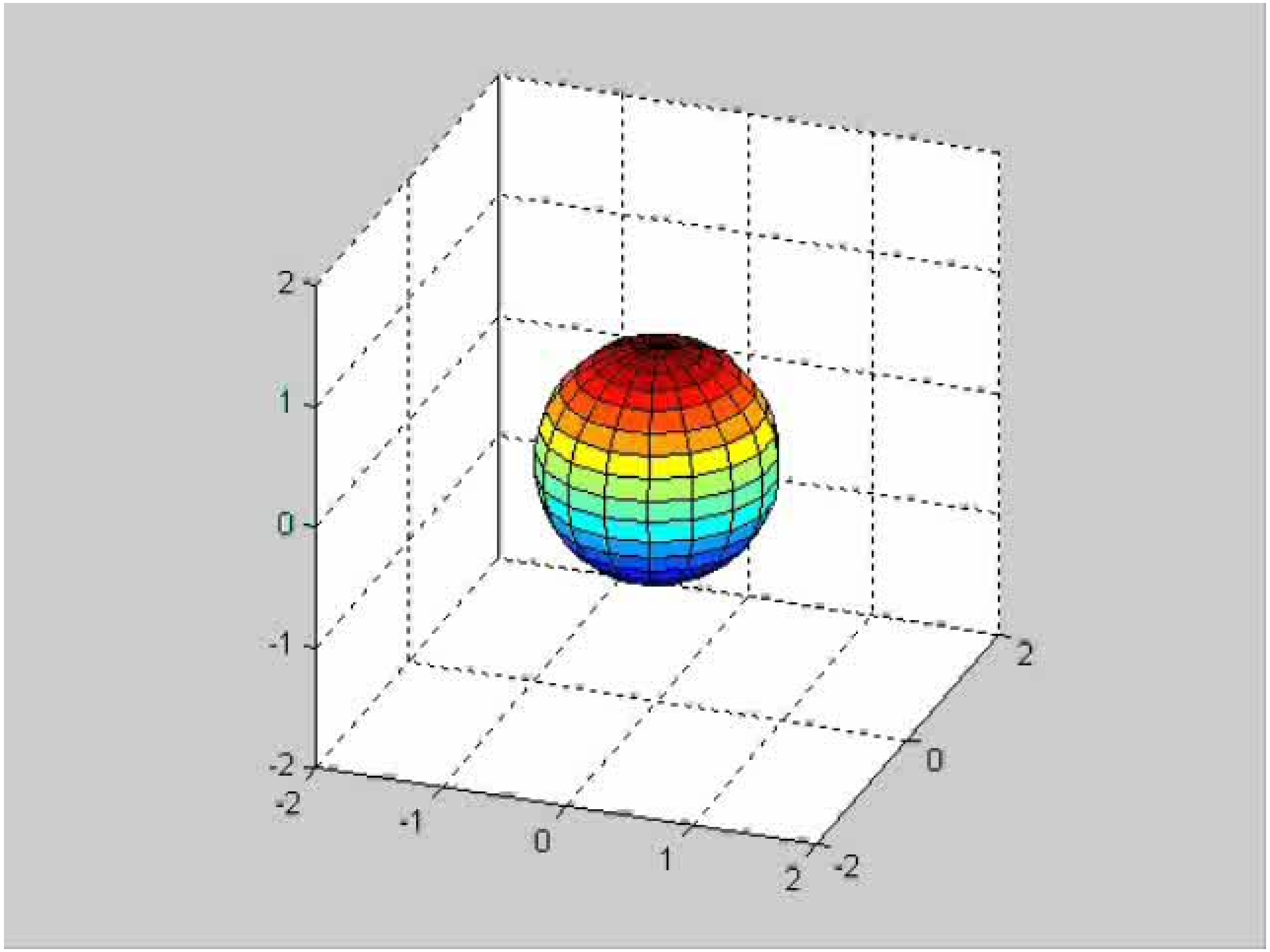
Discussion

The main result following from our calculations is the indication to a degenerate nature of formation of a singularity in unstable newtonian self-gravitating gaseous bodies.

Only pure spherical models can collapse to singularity, but any kind of nonsphericity leads to nonlinear stabilization of the collapse by a dynamic motion, and formation of regularly or chaotically oscillating body.

This conclusion is valid for all unstable equations of state, namely, for adiabatic with $\gamma < 4/3$.

In reality a presence of dissipation leads to damping of these oscillations, and to final collapse of nonrotating model, when total energy of the body is negative.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- Paper is submitted to MNRAS
- Thank you for your attention

Dynamic stabilization of non-spherical bodies against unlimited collapse

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ABSTRACT
We solve equations, describing in a simplified way a newtonian dynamics of a selfgravitating nonrotating spheroidal body after loss of stability. We obtain that contraction to singularity happens only in a pure spherical collapse, and deviations from the spherical symmetry stop the contraction by stabilising action of nonlinear nonspherical oscillations. A real collapse happens after damping of oscillations due to energy losses, shock wave formation or viscosity. Detailed analysis of the nonlinear oscillations is performed using Poincaré map construction. Regions of regular and chaotic oscillations are localized on this map.

Key words: gravitation – instabilities.