On the absence of stable periodic orbits in domains of separatrix crossings in non-symmetric slow-fast Hamiltonian systems

Anatoly Neishtadt[1,2] and Alexei Vasiliev[2]

[1] Dept. of Math. Sciences, Loughborough University, Loughborough, LE11 3TU, UK
[2] Space Research Institute, Profsoyuznaya 84/32, Moscow 117997, Russia

Abstract

We consider a 2 d.o.f. Hamiltonian system with one degree of freedom corresponding to fast motion and the other corresponding to slow motion. We assume that at frozen values of the slow variables there is a separatrix on the phase plane of the fast variables and there is a region in the phase space (the domain of separatrix crossings) where projections of phase points onto the plane of the fast variables repeatedly cross the separatrix in the process of evolution of the slow variables. Under rather general conditions we prove that there are no stable periodic trajectories of any prescribed period inside the domain of separatrix crossings, except maybe for periodic trajectories passing anomalously close to the saddle point.

We study a dynamical system with two typical scales of motion, the fast motion and the slow one. When investigating such a system, it is natural, as the first approximation, to "freeze" the slow motion and study the fast motion at constant values of the slow variables. We assume that the fast system thus obtained has an 8-shaped separatrix on its phase portrait. In the course of the slow evolution this separatrix slowly breaths, and phase trajectories of the system may cross it. It is a well-known fact that dynamics in the domain of separatrix crossings in such systems looks chaotic in computer simulations. However, only few rigorous results on this topic are obtained. In a series of recent papers, it was shown that if the fast system possesses additional symmetry, there are many stable periodic trajectories in this domain. Each stable periodic trajectory is surrounded by an island of stability, where dynamics is regular. In the present paper, we show that if the additional symmetry condition is not imposed onto the fast system, in general there are no stable periodic trajectories inside the domain of separatrix crossings, except maybe for periodic trajectories passing anomalously close to the saddle point.

1 Introduction

Many problems in theory of charged particles' motion, theory of propagation of short-wave excitations, and celestial mechanics can be reduced to analysis of 2 d.o.f. Hamiltonian systems with fast and slow variables (see, e.g. [1, 2, 3]). One degree of freedom corresponds to fast variables, and the other corresponds to slow variables. The typical ratio of time derivatives of the slow and the fast variables is of order $\varepsilon \ll 1$. To describe dynamics in such systems one can use the adiabatic approximation constructed as follows.

Consider the system for the fast variables at frozen values of the slow variables (we call it the fast system). This is a 1 d.o.f. Hamiltonian system involving the slow variables as parameters. Assume that for a range of values of the slow variables there is a region filled with closed trajectories on the phase portrait of the fast system. Then one can introduce "action-angle" variables in the fast system [4]. The "action" variable of the fast system is an adiabatic invariant (i.e. an approximate first integral) of the complete system: its value on a phase trajectory oscillates with amplitude $\sim \varepsilon$ on time intervals of order $\sim 1/\varepsilon$. To describe approximately variation of the slow variables, one should average the rates of their variation over the "angle" variable of the fast motion. This approximation of the real motion is called the adiabatic approximation [5]. The obtained 1 d.o.f. Hamiltonian system for the slow variables, involving the "action" variable of fast motion as a parameter, is called the slow system.

We shall consider the situation when there are separatrices on the fast system's phase portrait (see Figure 1), and there is a region in the phase space (the domain of separatrix crossings) where projections of phase points onto the plane of the fast variables repeatedly cross the separatrix in the process of evolution of the slow variables. In this case the described above construction of the adiabatic approximation needs a certain modification, and the formulated above assertions on the adiabatic approximation accuracy are not applicable. In particular, when a phase trajectory crosses a narrow neighborhood of the separatrix, the value of the adiabatic invariant undergoes a quasi-random jump of order $\varepsilon \ln \varepsilon$ [6]. Accumulation of such quasi-random jumps at multiple separatrix crossings results in chaotic dynamics of the system in the domain of separatrix crossings. In computer simulations, in many problems this domain looks like a region of dynamical chaos (see, e.g. [7, 8]).

Quite unexpectedly, however, it was shown that under additional symmetry conditions imposed on the fast system, in the domain of separatrix crossings there exist many stable periodic trajectories (in [9, 10], this result was obtained for Hamiltonian systems with one and a half degrees of freedom; in [11, 12], it was obtained for Hamiltonian systems with two degrees of freedom). Each one of these trajectories is surrounded by a stability island. The total measure of these islands does not tend to zero as ε tends to zero. The proof was based on analysis of Poincaré maps constructed with the use of results of [13, 14, 15, 16, 6, 17].

In this paper we consider a Hamiltonian system with two degrees of freedom without the additional symmetry conditions imposed. We prove that in general in the domain of the separatrix crossings there are no stable periodic trajectories of any prescribed period, except maybe for periodic trajectories passing anomalously close to the saddle point. A similar result for Hamiltonian systems with one and a half degrees of freedom was earlier obtained in [9]. Speaking very roughly, one can say that the result is due to the fact that jump in the adiabatic invariant in this case is of order $\varepsilon \ln \varepsilon$, and not ε like in the case considered in [11, 12]. We restrict our study to analysis of so called natural Hamiltonian systems, i.e. systems with Hamiltonian function $H = \frac{1}{2}g(x)y^2 + \frac{1}{2}\beta(x)p^2 + U(q,x)$ (see Section 2 for the used notations). For such Hamiltonian, the corresponding slow system possesses a symmetry that allows existence of periodic trajectories of the adiabatic approximation. For a generic Hamiltonian system with two degrees of freedom there are no periodic adiabatic trajectories, hence the question of stability for periodic trajectories of the exact system is not applicable in this case.

The paper has the following structure. In Section 2, we describe the system and discuss phase portraits. Section 3 is devoted to construction of adiabatic and improved adiabatic approximations. In Section 4, we describe two successive separatrix crossings and introduce parameters, characterizing each crossing. We write down the Poincaré map in Section 5 and prove the absence of stable periodic trajectories in Section 6. Existence of a large number, $\sim \varepsilon^{-1}|\ln\varepsilon|$, of periodic solutions in the considered range of parameters is proved in Appendix.

2 Properties of the system and main assumptions

Consider a 2 d.o.f. Hamiltonian system with the Hamiltonian function

$$H = \frac{1}{2}g(x)y^2 + \frac{1}{2}\beta(x)p^2 + U(q,x). \tag{1}$$

Here (p,q) and $(y, \varepsilon^{-1}x)$ are the pairs of canonically conjugated variables, $\varepsilon > 0$ is a small parameter. All the functions are assumed to be smooth enough, and the functions g and β are positive. The equations of motion have the form:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{y} = -\varepsilon\frac{\partial H}{\partial x}, \quad \dot{x} = \varepsilon\frac{\partial H}{\partial y}. \tag{2}$$

The variables p, q are called fast, while the variables y, x are called slow.

The system with Hamiltonian H at frozen values of the slow variables is called the fast system. It contains the value of x as a parameter. This is a natural system with one degree of freedom. Assume that the potential U is a function of q with two minima at all the considered values of x (see Figure 1, left). The potential U has the local maximum at the point $q = q_s(x)$.

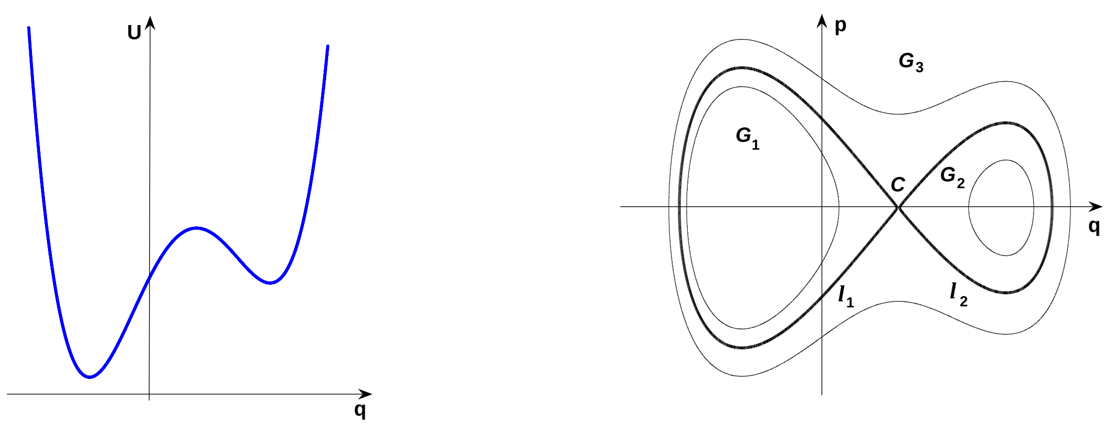

Figure 1: A non-symmetric double-well potential (left) and the plane of the fast variables (right).

The phase portrait of the fast system is shown in Figure 1, right. On this phase portrait, there exists a non-degenerate saddle point C and separatrices l_1, l_2 passing through this point. These separatrices divide the (p, q)-plane into domains $G_i = G_i(x)$, $i = 1, 2, 3$. Introduce the function

$$E = \frac{1}{2}\beta(x)p^2 + U(q, x) - U(q_s(x), x). \tag{3}$$

In the domain G_3 function E is positive, and in $G_{1,2}(x)$ it is negative. At the point C and on the separatrices $E = 0$.

Let $S_{1,2} = S_{1,2}(x)$ denote the area of $G_{1,2}(x)$ accordingly, $S_3 = S_3(x) = S_1(x) + S_2(x)$. In the unperturbed system, one can introduce canonical "action-angle" variables (I_i, φ_i) separately in each G_i. The "action" I_i is a function of p, q, x given by the formula $I_i = I_i(E, x)$, where $I_i(e, x)$ is an area bounded by a trajectory of the fast system, corresponding to $E = e$ in G_i, divided by 2π.

For an approximate description of motion in system (2) one can use the adiabatic approximation (see, e.g. [5]). In this approximation, while the projection of a phase point onto the fast plane (p, q) is in G_i, $I_i = \text{const}$ along the phase trajectory. Variation of the slow variables y, x is defined by a Hamiltonian system with Hamiltonian $H_{0,i}(I_i, y, x)$, where $H_{0,i}$ is the function H expressed in terms of I_i, y, x. Thus, $H_{0,i} = \frac{1}{2}g(x)y^2 + V_i(I_i, x)$, where $V_i(I_i, x)$ is the effective potential given by the function $\frac{1}{2}\beta(x)p^2 + U(q, x)$ expressed in terms of I_i, x. We shall consider the system on an energy level $H = h_0$, hence $H_{0,i} = h_0,\ i = 1, 2, 3$.

Introduce on the plane of variables (y, x) the uncertainty curve [3] $\Gamma(h_0) = \{y, x : \frac{1}{2}g(x)y^2 + U(q_s(x), x) = h_0\}$.

Motion in the adiabatic approximation can be described as follows (see Figure 2). Denote $\Theta_i = gy\partial S_i/\partial x$. As the system evolves, the area S_i varies at the speed of $\varepsilon\Theta_i$. An adiabatic trajectory, hence, can approach the separatrix and cross it. After crossing the separatrix, the motion continues in a different domain. Let the motion start at $t = 0$ at a point $M_0 = (p_0, q_0, y_0, x_0)$ and $(p_0, q_0) \in G_3(x_0)$. Let $I_3 = I^{(0)}$ at this point. The corresponding adiabatic trajectory is defined as follows. The part of this trajectory that corresponds to motion in G_3 is given by $I_3 = I_3(h_0 - \frac{1}{2}g(x)y^2 - U(q_s(x), x), x) = I^{(0)}$. Its projection onto the plane of the slow variables (= slow plane) is a curve $B_3(I^{(0)})$. Assume that it approaches the uncertainty curve Γ at a point (x_*, y_*) and $\Theta_i(x_*, y_*) > 0,\ i = 1, 2, 3$. (Let, for definiteness, $y_* > 0$; we assume, that there are no other intersections between this adiabatic trajectory and the uncertainty curve at $y > 0$.) This corresponds to the separatrix crossing on the plane of fast variables (=fast plane). After crossing the separatrix, the motion on the fast plane continues either in G_1 or in G_2. (A certain probability can be associated with each of these continuations, see [6].) If motion on the fast plane continues in domain G_i then on the plane of the slow variables, the motion is defined by the Hamiltonian $H_{0,i}(I_i, y, x)$ with $I_i = I_{i*} = S_i(x_*)/(2\pi)$. The projection of the corresponding part of the adiabatic trajectory onto the slow plane is a curve $B_i(I^{(0)})$ [we retain here the same argument $I^{(0)}$, because the value I_{i*} is uniquely defined by $I^{(0)}$]. Due to the symmetry of the system, the corresponding adiabatic trajectory approaches the uncertainty curve again at the point $(x_*, -y_*)$. After crossing the separatrix, the motion on the slow plane is defined by the Hamiltonian $H_{0,3}(I_3, y, x)$ with $I_3 = I^{(0)}$ and follows again the curve $B_3(I^{(0)})$. Thus, on the plane of the slow variables, the corresponding trajectory in the adiabatic approximation can be represented as a union of two segments: $B_3(I^{(0)})$, corresponding to the motion in G_3, and $B_i(I^{(0)})$, corresponding to the motion in G_i (see Figure 2). We assume that this trajectory belongs to a domain (a sort of an annulus, see

Figure 2) $\mathcal{D}_{3,i}$ on the plane of the slow variables, filled up with adiabatic trajectories of the system at $H_{0,i} = h_0$, and each of these trajectories possesses the same properties as the one described above. The values of I_3, I_i corresponding to these trajectories fill up segments Ξ_3, Ξ_i accordingly.

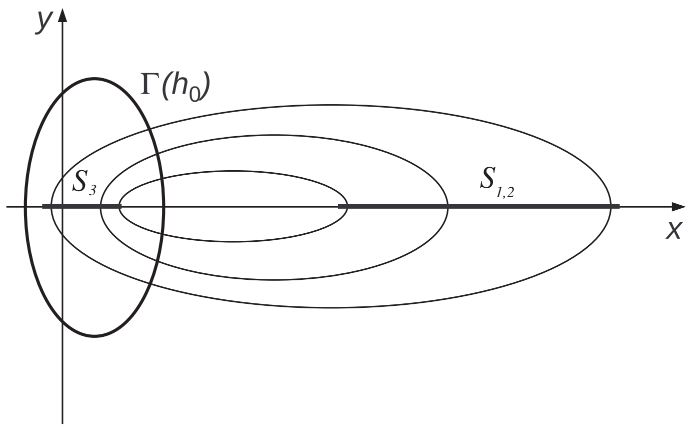

Figure 2: Schematic picture of motion on the slow plane.

The uncertainty curve divides $\mathcal{D}_{3,i}$ into two subdomains $\mathcal{D}_3$, $\mathcal{D}_i$. Consider a point $(y, x) \in \mathcal{D}_3$. The corresponding energy level line $H = h_0$ on the plane of the fast variables (p, q) belongs to the domain $G_3(x)$ and gives a trajectory of the fast system. For a point $(y, x) \in \mathcal{D}_i$, $i = 1, 2$, the corresponding energy level has connected component which is the trajectory of the fast system, belonging to the domain G_i. For a point $(y, x) \in \Gamma$ the energy level is a union of the separatrices $l_1(x), l_2(x)$ and the point $C(x)$.

3 Adiabatic and improved adiabatic approximations

In the fast system, action-angle variables I, $\varphi \bmod 2\pi$ are introduced separately in each domain G_i, $i = 1, 2, 3$ by a canonical transformation of variables. The corresponding generating function $W(I, q, x)$ contains x as a parameter [for brevity, we omit subscripts i]. We take this function in the form

$$W(I, q, x) = \int_{q_0(I,x)}^{q} \mathcal{P}(I, q', x)\mathrm{d}q', \tag{4}$$

where $\mathcal{P}$ is the value of p-variable along the trajectory with the prescribed value of action I. In G_1, G_2, $q_0(I, x)$ is the value of q at one of the two points where this trajectory crosses the axis $p = 0$; in G_1 we take the right one of these points, and in G_2 we take the left one. In G_3 we take $q_0(I, x) = q_s(x)$ and assume that at this point $\mathcal{P} > 0$. In the new variables the Hamiltonian has the form $H = H_0(I, y, x)$.

Now make a canonical transformation of variables $(p, q, y, x) \mapsto (\bar{I}, \bar{\varphi}, \bar{y}, \bar{x})$ with the generating function $\bar{y}\varepsilon^{-1}x + W(\bar{I}, q, x)$. The canonically conjugated pairs of variables are $(\bar{I}, \bar{\varphi})$ and $(\bar{y}, \varepsilon^{-1}\bar{x})$. Formulas for the transformation of variables are:

$$\bar{\varphi} = \partial W/\partial \bar{I}, \quad p = \partial W/\partial q, \quad \bar{x} = x, \quad y = \bar{y} + \varepsilon \partial W/\partial x. \tag{5}$$

In the new variables, Hamiltonian H has the form

$$H = H_0(\bar{I}, \bar{y}, \bar{x}) + \varepsilon H_1(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}) + \varepsilon^2 H_2(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}, \varepsilon), \tag{6}$$

where

$$H_1 = g\bar{y}\frac{\partial W}{\partial x}. \tag{7}$$

In the adiabatic approximation, dynamics is described by Hamiltonian H_0. In this approximation $\bar{I} = \text{const}$ along a phase trajectory.

One can also construct a canonical, close to identical, transformation of variables $(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}) \mapsto (J, \psi, \hat{y}, \hat{x})$ in order to make the terms of order ε in the Hamiltonian independent of phase (see, for example, [17]). In the new variables, the Hamiltonian takes the form:

$$\mathcal{H} = H_0(J, \hat{y}, \hat{x}) + \varepsilon\mathcal{H}_1(J, \hat{y}, \hat{x}) + \varepsilon^2\mathcal{H}_2(J, \psi, \hat{y}, \hat{x}, \varepsilon), \ \ \mathcal{H}_1 = \langle H_1 \rangle, \tag{8}$$

where the angular brackets denote averaging with respect to $\bar{\varphi}$. In fact, in G_1, G_2 the terms of order ε are absent because in these domains average value of H_1 over $\bar{\varphi}$ is zero due to the symmetry with respect to the axis $p = 0$.

In the improved adiabatic approximation, the dynamics is described by the Hamiltonian $H_0(J, Y, X) + \varepsilon\mathcal{H}_1(J, Y, X)$. In this approximation J is an integral of motion. With the accuracy of order ε^2, the following formula for J is valid (see [6]):

$$J = J(p, q, y, x) = I + \varepsilon u(p, q, y, x), \tag{9}$$

$$u = \frac{1}{2\pi} gy \int_0^T \left(\frac{T}{2} - t\right) \frac{\partial E}{\partial x}\,\mathrm{d}t. \tag{10}$$

The integral here is calculated along a phase trajectory of the fast system passing through the point (p, q); t is the time of motion along this trajectory starting from this point, T is the period of motion. Function J is the improved adiabatic invariant. In the complete system far from separatrices its value along a phase trajectory is constant with the accuracy $\mathrm{O}(\varepsilon^2)$ on time intervals of order ε^{-1}. Below, we use notation J_i for the improved adiabatic invariant in domain G_i.

4 Description of separatrix crossing

On the phase plane of the slow variables (y, x), the separatrix is represented by the curve $\Gamma(h_0)$ (see Figure 2). Fix the number ν, $\nu = 1$ or $\nu = 2$. We are interested in dynamics of phase points that are being captured into G_ν after first separatrix crossing. The domain $\mathcal{D}_{3,\nu}$ is divided by the uncertainty curve into domains $\mathcal{D}_3$ and $\mathcal{D}_\nu$. The axis $y = 0$ intersects the domain $\mathcal{D}_{3,\nu}$ along a segment $\mathcal{S}_3$ belonging to $\mathcal{D}_3$ and a segment $\mathcal{S}_\nu$, which belongs to $\mathcal{D}_\nu$ (see Figure 2). Points on the fast plane corresponding to $\mathcal{S}_3$ belong to G_3, and the points corresponding to $\mathcal{S}_\nu$ belong to G_ν. Fix an interval $\Xi_{3,0} \in \Xi_3$, endpoints of $\Xi_{3,0}$ and Ξ_3 are different.

Let the motion start at $t = 0$ at a point $M^{(0)} = (p^{(0)}, q^{(0)}, y^{(0)}, x^{(0)})$, such that $H(p^{(0)}, q^{(0)}, y^{(0)}, x^{(0)}) = h_0$ and $(\hat{y}^{(0)}, \hat{x}^{(0)}) \in \mathcal{S}_3$. Hence, $(p^{(0)}, q^{(0)}) \in G_3(x^{(0)})$. Let

$I_3 = I^{(0)} \in \Xi_{3,0},\ J_3 = J^{(0)},\ \psi = \psi^{(0)}$ at this point. In the (y,x)-plane, the part of the adiabatic trajectory belonging to $\mathcal{D}_3$ is $B_3(I^{(0)}) = \{y, x\ :\ (y,x) \in \mathcal{D}_3,\ H_{0,3}(I^{(0)}, y, x) = h_0\}$, the motion is described by a Hamiltonian system with Hamiltonian $H_{0,3}(I^{(0)}, y, x)$. Let (x_*, y_*) and $(x_*, -y_*)$ denote points of intersection of this trajectory and $\Gamma(h_0)$. Assume for definiteness that $y_* > 0$, $\Theta_{3*} \equiv \Theta_3(y_*, x_*) > 0$. Thus, a phase point of the slow system passes from $\mathcal{D}_3$ to $\mathcal{D}_\nu$ at the point (x_*, y_*), and at the point $(x_*, -y_*)$ it passes from $\mathcal{D}_\nu$ to $\mathcal{D}_3$. Introduce "slow time" variable $\tau = \varepsilon t$. Below τ_* denotes the slow time moment of the first separatrix crossing in this approximation. Let $I_{\nu*} = S_\nu(x_*)/(2\pi)$. After crossing the separatrix, adiabatic trajectory on the slow plane is curve $B_\nu(I^{(0)}) = \{y, x\ :\ (y,x) \in \mathcal{D}_\nu,\ H_{0,\nu}(I_{\nu*}, y, x) = h_0\}$. In the exact system, in the process of evolution the phase point on the plane of the slow variables approaches the curve $\Gamma(h_0)$, and accordingly, on the plane (p,q) it approaches the separatrix. For such a phase point, the last crossing of Cp-axis in G_3 (see Figure 1) occurs near C at time $\tau = \tau_* + \mathrm{O}(\varepsilon \ln \varepsilon)$. Assuming that at the point of this crossing $E = h^{(0)}$, we introduce $\eta^{(0)} = h^{(0)}/(\varepsilon \Theta_{3*})$.

After the separatrix crossing the phase point in the plane of slow variables $(\hat{y}, \hat{x})$ moves towards $\mathcal{S}_\nu$. When it crosses $\mathcal{S}_\nu$, the projection of the phase point onto the fast plane is deep inside region G_ν. Denote the value of J_ν at this time moment as $J^{(1)}$.

Then the phase point again starts approaching the separatrix. In the adiabatic approximation, projection of the phase point onto the plane of the slow variables follows the curve B_ν. Let τ^ν_{**} denote the slow time moment of passing through the point $(x_*, -y_*)$ in the adiabatic approximation. At $\tau = \tau^{(\nu)}_{**} + \mathrm{O}(\varepsilon \ln \varepsilon)$ the phase point crosses Cq-axis in G_ν near point C for the last time before entering G_3. Denote $\Theta_{\nu*} \equiv \Theta_\nu(y_*, x_*) > 0$. Assuming that at the point of this crossing $E = h^{(1)}$, we introduce $\eta^{(1)} = -h^{(1)}/(\varepsilon \Theta_{\nu*})$. [We have taken into account that $\Theta_\nu(-y_*, x_*) = -\Theta_\nu(y_*, x_*)$.]

After crossing $\Gamma(h_0)$, the projection of the phase point onto the plane $(\hat{y}, \hat{x})$ crosses again the segment $\mathcal{S}_3$. Let $J^{(2)},\ \psi^{(2)}$ denote values of $J_3,\ \psi$ at this time moment. Then the phase point approaches the separatrix again, crosses it and gets captured into domain G_l, $l = 1$ or $l = 2$. Let $E = h^{(2)}$ at the last crossing of Cp-axis in G_3 near C before this crossing. This crossing occurs at time $\tau = \tau_* + T_0^{(\nu)} + \mathrm{O}(\varepsilon \ln \varepsilon)$, where $T_0^{(\nu)}$ is the slow time period of motion along the trajectory $B(I^{(0)}) = B_3(I^{(0)}) \bigcup B_\nu(I^{(0)})$ in the adiabatic approximation. We introduce $\eta^{(2)} = h^{(2)}/(\varepsilon \Theta_{3*})$.

5 The return map

In this section, we introduce the return map describing the dynamics of the system in the domain of separatrix crossings.

In the energy level $H = h_0$ segment $\mathcal{S}_3$ is represented by a piece of two-dimensional surface $\{p, q, y, x\ :\ H(p,q,y,x) = h_0, (\hat{y}, \hat{x}) \in \mathcal{S}_3\}$. This piece can be parameterized by variables J_3, ψ. The corresponding Poincaré return map $M\ :\ (J^{(0)}, \psi^{(0)}) \to (J^{(2)}, \psi^{(2)})$ produced by trajectories that pass through G_ν is symplectic. Its stationary points correspond to periodic orbits of the original problem, of period approximately equal to T_0^ν/ε. It is convenient to study these stationary points using a different set of variables, namely to consider map $\hat{M}\ :\ (J^{(0)}, \eta^{(0)}, \nu) \to (J^{(2)}, \eta^{(2)}, l)$.

The map $\hat{M}$ is a composition of two maps: $\hat{M} = M^{(2)} \circ M^{(1)}$,

$$M^{(1)} : (J^{(0)}, \eta^{(0)}, \nu) \to (J^{(1)}, \eta^{(1)}, \nu)\,, \quad M^{(2)} : (J^{(1)}, \eta^{(1)}, \nu) \to (J^{(2)}, \eta^{(2)}, l)\,.$$

The results of [6, 17] give the following formulas for $M^{(k)}$. Suppose that

$$\eta^{(0)} \in \left[c_1^{-1}, \frac{\Theta_{\nu *}}{\Theta_{3*}} - c_1^{-1}\right], \quad \eta^{(1)} \in [c_1^{-1}, 1 - c_1^{-1}],$$
$$\eta^{(2)} \in \left[c_1^{-1}, \frac{\Theta_{l*}}{\Theta_{3*}} - c_1^{-1}\right], \tag{11}$$

where c_1 is a positive constant, which can be chosen arbitrarily large. Then

$$2\pi J^{(1)} = S_\nu(x_*) + \varepsilon a_* \frac{\Theta_{\nu *}}{\Theta_{3*}}(\Theta_{3*} - 2\Theta_{\nu *})\left(\frac{1}{2} - \frac{\Theta_{3*}}{\Theta_{\nu *}}\eta^{(0)}\right)\ln\varepsilon + O(\varepsilon), \tag{12}$$

$$\eta^{(1)} = \left\{\frac{\Theta_{3*}}{\Theta_{\nu *}}\eta^{(0)} + \varepsilon^{-1}\Phi_1^\nu(J^{(1)}) + O(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3})\right\}, \tag{13}$$

$$2\pi J^{(2)} = S_3(\tilde{x}_*) - \varepsilon a_*(\Theta_{3*} - 2\Theta_{\nu *})(\eta^{(1)} - \frac{1}{2})\ln\varepsilon + O(\varepsilon), \tag{14}$$

$$\eta^{(2)} = \left\{\frac{\Theta_{\nu *}}{\Theta_{3*}}\eta^{(1)} - (1 - \delta_{\nu,l})\frac{\Theta_{\nu *}}{\Theta_{3*}} + \varepsilon^{-1}\Phi_2(J^{(2)}) + O(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3})\right\}, \tag{15}$$

$$l = \nu, \quad \text{if} \quad 0 < \left\{\frac{\Theta_{\nu *}}{\Theta_{3*}}\eta^{(1)} + \varepsilon^{-1}\Phi_2(J^{(2)})\right\} < \frac{\Theta_{\nu *}}{\Theta_{3*}}, \tag{16}$$

$$l \neq \nu, \quad \text{if} \quad \frac{\Theta_{\nu *}}{\Theta_{3*}} < \left\{\frac{\Theta_{\nu *}}{\Theta_{3*}}\eta^{(1)} + \varepsilon^{-1}\Phi_2(J^{(2)})\right\} < 1. \tag{17}$$

Here $\{\cdot\}$ denotes the fractional part, $a_* = a(x_*)$. For a one has $a = 1/\sqrt{-d}$, where d is the Hessian of E at the point C. In (13), (15),

$$\Phi_1^\nu(J) = \frac{1}{2\pi}\int_{\tau_-^{(\nu)}}^{\tau_+^{(\nu)}} (\omega_0^{(\nu)}(J, Y_\nu(\tau), X_\nu(\tau))\, d\tau, \tag{18}$$

$$\Phi_2(J) = \frac{1}{2\pi}\int_{\tau_+^{(3)}}^{\tau_-^{(3)}} (\omega_0^{(3)}(J, Y_3(\tau), X_3(\tau) + \varepsilon\omega_1^{(3)}(J, Y_3(\tau), X_3(\tau)))\, d\tau. \tag{19}$$

Here $\omega_0^{(j)} = \partial H_0/\partial J$, $\omega_1^{(j)} = \partial\mathcal{H}_1/\partial J$, with H_0, $\mathcal{H}_1$ calculated in region G_j, and (Y_j, X_j) is a solution of Hamiltonian system with Hamiltonian $H_0(J, y, x) + \varepsilon\mathcal{H}_1(J, y, x)$ on the energy level $H_0 = h_0$. Values $\tau_\pm^{(j)}$ are the slow time moments when the phase point corresponding to this solution arrives to the separatrix. At these moments $S_j(X_j) = 2\pi J$. [One can take any of such solutions, they differ by a time shift which does not change the value of the integrals in (18), (19)]. In (15), $\delta_{\nu,l}$ is the Kroneker symbol. The value $\tilde{x}_*$ in (14) is defined by $S_\nu(\tilde{x}_*) = 2\pi J^{(1)}$.

Equations (12), (14) follow directly from the formula for a jump of adiabatic invariant at a separatrix [6] for the case with the assumed symmetry properties. Equations (13), (15)-(17) are derived from the result of [17] in the following way.

Let $h_\nu^{(0)}$ be the value of E when the phase point crosses the axis $p = 0$ near C for the first time after entering G_ν. We have (see [6]): $h_\nu^{(0)} = h^{(0)} - \varepsilon\Theta_{\nu*} + O(\varepsilon^{3/2})$. Introduce value $\xi_\nu^{(0)} = |h_\nu^{(0)}/(\varepsilon\Theta_{\nu*})|$. This value is related to $\eta^{(0)}$ as

$$\xi_\nu^{(0)} = 1 - \frac{\Theta_{3*}}{\Theta_{\nu*}}\eta^{(0)} + O(\sqrt{\varepsilon}). \tag{20}$$

According to [17],

$$\xi_\nu^{(0)} + \eta^{(1)} = \varepsilon^{-1}\Phi_1^\nu(J^{(1)}) + O(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}) \mod 1. \tag{21}$$

Equation (13) then immediately follows from (21) and (20).

Let $h_3^{(1)}$ be the value of E when the phase point crosses the axis $q = 0$ near C for the first time after exit from G_ν. We have: $h_3^{(1)} = h^{(1)} + \varepsilon\Theta_{\nu*} + O(\varepsilon^{3/2})$. At this time moment the phase point is in G_3 on positive or negative part of axis Cp. Let $\tilde{h}^{(2)}$ denote the value of E when the phase point crosses the same part of the axis Cp for the last time before entering G_l. Introduce values $\xi_3^{(1)} = h_3^{(1)}/(\varepsilon\Theta_{3*})$ and $\tilde{\eta}^{(2)} = \tilde{h}^{(2)}/(\varepsilon\Theta_{3*})$. If $0 < \tilde{\eta}^{(2)} < 1 - \Theta_{\nu*}/\Theta_{3*}$, we have $l \neq \nu$; if $1 - \Theta_{\nu*}/\Theta_{3*} < \tilde{\eta}^{(2)} < 1$, we have $l = \nu$.

Consider first the case $l \neq \nu$. In this case $\tilde{h}^{(2)} = h^{(2)}$ and $\tilde{\eta}^{(2)} = \eta^{(2)}$. From the definitions of $\xi_3^{(1)}$ and $h_3^{(1)}$ we find

$$\xi_3^{(1)} = \frac{\Theta_{\nu*}}{\Theta_{3*}}(1 - \eta^{(1)}) + O(\sqrt{\varepsilon}). \tag{22}$$

According to [17],

$$\xi_3^{(1)} + \eta^{(2)} = \varepsilon^{-1}\Phi_2(J^{(2)}) + O(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}) \mod 1. \tag{23}$$

From (23), (22) we obtain (15) with $l \neq \nu$. The condition (17) then follows from (15) and the condition $0 < \eta^{(2)} < 1 - \Theta_{\nu*}/\Theta_{3*}$.

Now proceed to the case $l = \nu$. Introduce $\tilde{h}_3^{(1)}$ as the value of E when the phase trajectory crosses the Cp-axis for the second time after exit from G_ν. We have: $\tilde{h}_3^{(1)} = h_3^{(1)} + \varepsilon(\Theta_{3*} - \Theta_{\nu*}) + O(\varepsilon^{3/2})$. Introduce $\tilde{\xi}_3^{(1)} = \tilde{h}_3^{(1)}/(\varepsilon\Theta_{3*})$. From the definitions of $\tilde{\xi}_3^{(1)}$ and $\tilde{h}_3^{(1)}$ we find

$$\tilde{\xi}_3^{(1)} = 1 - \frac{\Theta_{\nu*}}{\Theta_{3*}}\eta^{(1)} + O(\sqrt{\varepsilon}). \tag{24}$$

According to [17],

$$\tilde{\xi}_3^{(1)} + \eta^{(2)} = \varepsilon^{-1}\Phi_2(J^{(2)}) + O(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}) \mod 1. \tag{25}$$

From (25), (24) we obtain (15) with $l = \nu$. In this case we have $\eta^{(2)} = \tilde{\eta}^{(2)} - 1 + \Theta_{\nu*}/\Theta_{3*} + O(\sqrt{\varepsilon})$. The condition (16) then follows from (15) and the condition $1 - \Theta_{\nu*}/\Theta_{3*} < \tilde{\eta}^{(2)} < 1$.

6 Absence of stable periodic trajectories

In Appendix, we prove that in the domain defined by $I^{(0)} \in \Xi_{3,0}$ and conditions (11), the system has many, of order $\varepsilon^{-1}|\ln\varepsilon|$, periodic trajectories. In this Section, we prove that all periodic trajectories in this domain and under these conditions are linearly unstable. Let the map $\hat{M}$ have a fixed point $J^{(0)} = J_0$, $\eta^{(0)} = \eta_0$. Near this fixed point the map can be linearized. The determinant of the Jacobi matrix for $\hat{M}$ at (J_0, η_0) equals 1. The linear stability condition for the fixed point is $|\mathrm{trace}\,\hat{M}| < 2$, where $\mathrm{trace}\,\hat{M}$ denotes trace of the Jacobi matrix for $\hat{M}$. On the other hand, formulas (12-15) imply that the trace of the Jacobi matrix at (J_0, η_0) equals

$$\mathrm{trace}\,\hat{M} = B(\ln\varepsilon)^2 + O(\ln\varepsilon), \tag{26}$$

where

$$B = \frac{a_*^2}{4\pi^2}(\Theta_{3*} - 2\Theta_{\nu*})^2 \left.\frac{\partial\bar{\Phi}_1^\nu(J)}{\partial J}\right|_{J=J_0-S_{\bar{\nu}}(x_*)/(2\pi)} \left.\frac{\partial\bar{\Phi}_2(J)}{\partial J}\right|_{J=J_0}. \tag{27}$$

Here $\bar{\nu} = 1$ if $\nu = 2$ and vice versa. Functions $\bar{\Phi}_1^\nu$ and $\bar{\Phi}_2(J)$ are "roughened" Φ_1^ν and $\Phi_2(J)$, accordingly:

$$\bar{\Phi}_1^\nu(J) = \frac{1}{2\pi}\int_{\tau_*}^{\tau_{**}^{(\nu)}} \omega_0^{(\nu)}(J, y_\nu(\tau), x_\nu(\tau))\,\mathrm{d}\tau\,, \tag{28}$$

$$\bar{\Phi}_2(J) = \frac{1}{2\pi}\int_{\tau_{**}^{(\nu)}}^{\tau_*+T_0^{(\nu)}} \omega_0^{(3)}(J, y_3(\tau), x_3(\tau))\,\mathrm{d}\tau. \tag{29}$$

In these integrals, $(y_j(\tau), x_j(\tau))$ is a solution of the Hamiltonian system with Hamiltonian $H_{0,j}(J, y, x)$ on the energy level $H_{0,j} = h_0$. The integration limits are also calculated in the adiabatic approximation (see notations in Section 4). Note that value of r.h.s. in (29) does not depend on ν.

We assume that $B \neq 0$. Thus, $|\mathrm{trace}\,\hat{M}| \to \infty$ as $\varepsilon \to 0$. Therefore, the linear stability condition $|\mathrm{trace}\,\hat{M}| < 2$ cannot be satisfied at small enough ε, and the fixed point is unstable. Moreover, a periodic point of $\hat{M}$ of any prescribed period is in general unstable. Indeed, each iteration of $\hat{M}$ produces a factor of order $(\ln\varepsilon)^2$ in the expression for the trace. Consider a periodic trajectory that starts at a point (J_0, η_0) in G_3 and returns to its starting point after being captured k times in G_ν and n times in G_l, $l \neq \nu$. Accordingly, the map $\hat{M}$ has a periodic point (J_0, η_0) of period $k+n$. The expression for the trace of the Jacobian matrix for $\hat{M}^{k+n}$ at (J_0, η_0) is:

$$\mathrm{trace}\,\hat{M}^{k+n} = B_{(k,n)}(\ln\varepsilon)^{2(k+n)} + O((\ln\varepsilon)^{2(k+n)-1}), \tag{30}$$

with

$$B_{(k,n)} = \frac{a_*^{2(k+n)}}{(2\pi)^{2(k+n)}}(\Theta_{l*} - \Theta_{\nu*})^{2(k+n)} \left(\frac{\partial\bar{\Phi}_1^\nu}{\partial J}\right)^k \left(\frac{\partial\bar{\Phi}_1^l}{\partial J}\right)^n \left(\frac{\partial\bar{\Phi}_2}{\partial J}\right)^{k+n}, \tag{31}$$

where the partial derivatives of $\bar{\Phi}_1^\nu$, $\bar{\Phi}_1^l$, and $\bar{\Phi}_2$ are calculated at $J = J_0 - S_l(x_*)/(2\pi)$, $J = J_0 - S_\nu(x_*)/(2\pi)$, and $J = J_0$, accordingly.

If the system possesses additional symmetry, such that $\Theta_1 = \Theta_2$, the value B in (26) is zero, and the periodic trajectory may be linearly stable. Indeed, existence of linearly stable periodic trajectories in the case of a symmetric double-well potential was proved in [12] (see also [11], where a more general case is considered). It can be seen from (26),(31) that for smaller values of asymmetry $|\Theta_1 - \Theta_2|$ one should take a smaller value of ε to make sure that the periodic trajectory is unstable. An estimate for the corresponding value of ε can be obtained as follows. The main terms in expressions (12),(14) for $\hat{M}$ are proportional to $\varepsilon \ln\varepsilon |\Theta_1 - \Theta_2|$, while the next term is of order ε (see [6]). Thus, the term containing the asymmetry prevails, if ε is of order $\exp(-\text{const}/|\Theta_1 - \Theta_2|)$ or smaller.

Summarizing, we can say the following. In the domain defined by $I^{(0)} \in \Xi_{3,0}$ and conditions (11), for any prescribed period can be found a small enough ε, such that the considered system does not have stable periodic solutions of this period and smaller. The condition $I^{(0)} \in \Xi_{3,0}$ implies that the periodic solution does not come too close to the boundary of the domain of separatrix crossings. Conditions (11) mean that the periodic solution does not pass anomalously close to the saddle point.

Acknowledgements

The work was supported in part by grants RFBR 06-01-00117 and NSh-1312.2006.1.

Appendix. Existence of periodic solutions

In this Appendix, we prove that map $\hat{M}$ has many, of order ε^{-1}, stationary points in the domain defined by $I^{(0)} \in \Xi_{3,0}$ and conditions (11). In the proof, we use the way of argument similar to one developed in [9, 10].

Introduce the following notations:

$$\hat{J}^{(s)} = \varepsilon\zeta^{(s)}, \quad s = 0, 1, 2; \tag{32}$$

$$\varepsilon^{-1}\frac{\mathrm{d}\Phi_1^{\nu}(\varepsilon\zeta)}{\mathrm{d}\zeta} = \gamma_1(\varepsilon\zeta), \qquad \varepsilon^{-1}\frac{\mathrm{d}\Phi_2(\varepsilon\zeta,\varepsilon)}{\mathrm{d}\zeta} = \gamma_2(\varepsilon\zeta,\varepsilon)$$

Suppose that $(\zeta^{(0)}, \eta^{(0)}) = (\zeta, \eta)$ is a stationary point of map $\hat{M}$ with $l = \nu$. Thus, $\zeta^{(2)} = \zeta$, $\eta^{(2)} = \eta$ and from (12)-(15) we obtain the following equations (for brevity, the subscript $*$ is omitted):

$$\eta^{(1)} = 1 - \frac{\Theta_3}{\Theta_\nu}\eta + O(\varepsilon^{1/2}) \tag{33}$$

$$\eta^{(1)} - \frac{\Theta_3}{\Theta_\nu}\eta - \gamma_1\left(\varepsilon\zeta - \frac{S_{\bar\nu}}{2\pi}\right)\left(a\frac{\Theta_\nu}{\Theta_3}(\Theta_{\bar\nu} - \Theta_\nu)\left(\frac{1}{2} - \frac{\Theta_3}{\Theta_\nu}\eta\right)\ln\varepsilon + A(\eta)\right)$$
$$= \varepsilon^{-1}\Phi_1^{\nu}\left(\varepsilon\zeta - \frac{S_{\bar\nu}}{2\pi}\right) + O(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}) \bmod 1 \tag{34}$$

$$\eta - \frac{\Theta_\nu}{\Theta_3}\eta^{(1)} = \varepsilon^{-1}\Phi_2(\varepsilon\zeta,\varepsilon) + O(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}) \bmod 1. \tag{35}$$

Here function $A(\eta)$ represents the terms of order ε in the formula for the jump in the improved adiabatic invariant at the separatrix crossing (see [6]). Here we do not need the explicit expression for this function.

Below, we look for solutions of (33)-(35) neglecting the error terms. Thus obtained set of equations differs from the exact set (33)-(35) by terms $O(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3})$ that can be differentiated without changing their order of smallness. We find non-degenerate solutions of the set (33)-(35) without the error terms. According to the implicit function theorem, for small enough ε, each of these solutions corresponds to a solution of the exact system.

We assume that there exists at least one point I such that

$$\gamma_1(I - S_{\bar\nu}/(2\pi))\gamma_2(I,0) \neq 0. \tag{36}$$

Therefore, there is an interval of values of I, where (36) is valid. Below, we consider only values of $\varepsilon\zeta$ that belong to the interval defined by (36).

Equation (33) defines in the first approximation a segment $L(\varepsilon\zeta)$ of a straight line on the plane $(\eta, \eta^{(1)})$. The endpoints of the segment are $(0,1)$ and $(\Theta_\nu/\Theta_3, 0)$.

Consider the points $(\eta, \eta^{(1)}) \in L(\varepsilon\zeta)$. Equations (33)-(35) can be interpreted geometrically as follows. Let the curve $\Lambda(\varepsilon\zeta) \subset \mathbb{T}^2 = \{(s_1 \bmod 1, s_2 \bmod 1)\}$ be the image of $L(\varepsilon\zeta)$ under the mapping

$$(\eta, \eta^{(1)}) \mapsto (s_1, s_2)$$

$$= \left(1-\frac{2\Theta_3}{\Theta_\nu}\eta-\gamma_1\left(\varepsilon\zeta-\frac{S_{\bar\nu}}{2\pi}\right)\left(a\frac{\Theta_\nu}{\Theta_3}(\Theta_{\bar\nu}-\Theta_\nu)\left(\frac{1}{2}-\frac{\Theta_3}{\Theta_\nu}\eta\right)\ln\varepsilon+A(\eta)\right),\, 2\eta-\frac{\Theta_\nu}{\Theta_3}\right) \tag{37}$$

defined by left hand sides of (34) and (35). Consider also the point

$$\varepsilon^{-1}\Phi(\varepsilon\zeta,\varepsilon) = \left(\varepsilon^{-1}\Phi_1^\nu\left(\varepsilon\zeta - \frac{S_{\bar\nu}}{2\pi}\right) \bmod 1, \quad \varepsilon^{-1}\Phi_2(\varepsilon\zeta,\varepsilon) \bmod 1\right) \tag{38}$$

on $\mathbb{T}^2$. As ζ varies, the curve $\Lambda(\varepsilon\zeta)$ moves slowly, at the speed of order $\varepsilon\ln\varepsilon$ on the torus. On the other hand, the point $\varepsilon^{-1}\Phi$ is moving fast, while its velocity vector $\gamma(\varepsilon\zeta,\varepsilon) = (\gamma_1(\varepsilon\zeta)\,, \gamma_2(\varepsilon\zeta,\varepsilon))$ varies slowly. Solutions of system (33)-(35) correspond to values of "time" ζ when the point $\varepsilon^{-1}\Phi(\varepsilon\zeta,\varepsilon)$ crosses the curve $\Lambda(\varepsilon\zeta)$.

One can see from (37) that $\Lambda(\varepsilon\zeta)$ is almost parallel to the meridian of the torus that corresponds to s_1. The length of $\Lambda(\varepsilon\zeta)$ is of order $|\ln\varepsilon|$. Hence, it makes $\sim |\ln\varepsilon|$ turns around the torus.

It follows from assumption (36), that the winding of the torus defined by (38) is transversal to Λ. The interval defined by (36) corresponds to an interval of "time" ζ of length more than c_2^{-1}/ε (c_2 and C_1 below are positive constants). On this "time" interval there are more than $C_1^{-1}|\ln\varepsilon|/\varepsilon$ pairs (ζ_r, η_r) such that the point $\varepsilon^{-1}\Phi(\varepsilon\zeta,\varepsilon)$ crosses transversally the moving curve $\Lambda(\varepsilon\zeta)$ at $\zeta = \zeta_r$ and $\eta = \eta_r$. Each intersection point corresponds to a stationary point of the return map. Each of these stationary points, in turn, corresponds to a periodic orbit of the original system.

References

[1] J. Büchner and L. M. Zelenyi, "Regular and chaotic charged particle motion in magnetotaillike field reversals, 1, Basic theory of trapped motion," J. Geophys. Res. **94**(A9), 11821-11842 (1989).

[2] A. V. Gurevich and E. E. Zedilina, *Long distance propagation of HF radio waves* (Springer-Verlag, Berlin, 1985).

[3] J. Wisdom, "A perturbative treatment of motion near the 3/1 commensurability," Icarus **63**(2), 272-289 (1985).

[4] V. I. Arnold, *Mathematical methods of classical mechanics* (Springer-Verlag, New York, 1978).

[5] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics (Dynamical Systems III. Encyclopedia of Mathematical Sciences)*, 3rd edition (Springer-Verlag, New York, 2006).

[6] A. I. Neishtadt, "On the change in the adiabatic invariant on crossing a separatrix in systems with two degrees of freedom," PMM USSR **51**, 586-592 (1987).

[7] J. R. Cary, D. S. Bruhwiller, "Diffusion of particles in a slowly modulated wave," Physica D **40**, 265 (1989).

[8] A. I. Neishtadt and V. V. Sidorenko, "Wisdom system: dynamics in the adiabatic approximation," Celest. Mech. and Dynam. Astron. **90**, 307-330 (2004).

[9] A. I. Neishtadt, V. V. Sidorenko, and D. V. Treschev, "Stable periodic motions in the problem of passage through a separatrix," Chaos **7**, 2-11 (1997).

[10] A. I. Neishtadt, V. V. Sidorenko, and D. V. Treschev, "On stability islands in the domain of separatrix crossings," In: *Nonlinear Mechanics* eds. V. M. Matrosov, V. V. Rumyantsev *et al* (Moscow: Fizmatlit), 192-203 (2001) (in Russian).

[11] A. A. Vasiliev, A. I. Neishtadt, C. Simó, and D. V. Treschev, Stability islands in domains of separatrix crossings in slow-fast Hamiltonian systems, Proc. of the V. A. Steklov Math. Inst., 2007, to appear; preprint math.DS/0611468

[12] A. I. Neishtadt, C. Simó, D. V. Treschev, and A. A. Vasiliev, Periodic orbits and stability islands in chaotic seas created by separatrix crossings in slow-fast systems, Discrete and Continuous Dynamical Systems, 2007, to appear.

[13] A. V. Timofeev, "On the constancy of an adiabatic invariant when the nature of the motion changes," Sov. Phys. JETP **48**, 656-659 (1978).

[14] A. I. Neishtadt, "Change of an adiabatic invariant at a separatrix," Sov.J.Plasma Phys. **12**, 568-573 (1986).

[15] J. R. Cary, D. F. Escande, and J. Tennyson, "Adiabatic invariant change due to separatrix crossing," Phys. Rev. **A34**, 4256-4275 (1986).

[16] J. R. Cary and R. T. Skodje, "Phase change between separatrix crossings," Physica D **36**, 287-316 (1989).

[17] A. I. Neishtadt and A. A. Vasiliev, "Phase change between separatrix crossings in slow-fast Hamiltonian systems," Nonlinearity **18**, 1393-1406 (2005).