Phase change between separatrix crossings in slow-fast Hamiltonian systems

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Abstract.

We consider a Hamiltonian system with slow and fast motions, one degree of freedom corresponding to fast motion, and the other degrees of freedom corresponding to slow motion. Suppose that at frozen values of the slow variables there is a non-degenerate saddle point and a separatrix on the phase plane of the fast variables. In the process of variation of the slow variables, the projection of a phase trajectory onto phase plane of the fast variables may repeatedly cross the separatrix. These crossings are described by the crossing parameter called pseudo-phase. We obtain an asymptotic formula for the pseudo-phase dependence on the initial conditions, and calculate change of the pseudo-phase between two subsequent separatrix crossings.

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1. Introduction

In this paper we consider a slow-fast Hamiltonian system (also called a system with slow and fast motions). In such a system one pair of canonical variables is changing at a typical rate of order one, and the other pairs are changing at a rate of order ε, where $0 < \varepsilon \ll 1$ is a small parameter. The averaging method (see, e.g. [1]) can be used for approximate description of dynamics in such systems. The "action" variable I introduced on the phase plane of the fast variables is an adiabatic invariant of the system. This means that under certain conditions I is preserved along phase trajectories of the system with an accuracy of order ε on time intervals of order ε^{-1}. An essential condition for applicability of the averaging method is that the frequency of motion on the fast phase plane is indeed of order one. This condition, however, is violated at separatrices that may exist on the fast phase plane. In the process of evolution of the slow variables separatrices slowly move. A phase trajectory may cross a separatrix, and hence, the frequency may vanish. Note that generally this phenomenon takes place for a measure of initial conditions of order one, and thus cannot be neglected. An estimate of accuracy of conservation of adiabatic invariance at separatix crossing was obtained in [2, 3].

Crossing of a separatrix results in a jump of the adiabatic invariant. An asymptotic formula for this jump was obtained in [4, 5, 6], in the case of explicit slow time dependence (this case can be reduced to the case of a slow-fast system with two degrees of freedom by introducing time as a new phase variable). It was shown that this jump depends on a value of so called pseudo-phase. This latter belongs to the interval $(0, 1)$ and strongly depends on initial conditions. If a phase trajectory crosses the separatrix repeatedly, accumulation of jumps, as one can see in numeric experiments, results in a diffusive-like behaviour of the adiabatic invariant. Dynamics in the region of separatrix crossings looks chaotic. However, not so many rigorous results were obtained on this dynamics. Existence of Smale-horseshoe-type dynamics on a set of zero measure in this region (the measure of the region itself is of order one) was shown in [7].

A formula for the change of the pseudo-phase between two subsequent separatrix crossings in the system with explicit slow time dependence was obtained in [8], [9], for the case of symmetric separatrix crossing this formula is presented in [10]. Together with the formula for the change of the adiabatic invariant [4, 5, 6], it can be used to define a mapping describing dynamics in the region swept by slowly moving separatrix on the phase plane. Under additional symmetry conditions, and when slow dependence of the Hamiltonian on time is periodic, this mapping was used in [11, 12] to prove existence in this region of stable periodic phase trajectories of period equal to the period of the Hamiltonian and stability islands surrounding these trajectories. Quite unexpectedly, it was found that there are a number of order ε^{-1} periodic trajectories surrounded by stability islands of measure of order ε. Hence, the total measure of these islands of stability is a value of order one, though small. In [9], it was shown that measure of a stability island in the region of separatrix crossings cannot exceed a value of order ε. An analog of asymptotic formulae for jumps of adiabatic invariant at separatrix crossings and variation of pseudo-phases in a class of dissipative systems is given in [13, 14].

A formula for the jump of adiabatic invariant in an autonomous slow-fast system with two degrees of freedom was obtained in [15]. In the present paper we derive a formula for the change of the pseudo-phase between two subsequent separatrix crossings in such a system. This formula makes it possible to describe dynamics in the region of separatrix crossings with the use of a mapping, similar to [11]. Systems of this kind describe various problems in mechanics and physics. Here we mention a restricted planar elliptic three body problem (Sun, Jupiter, and an asteroid) near a 3:1 resonance [16, 15] and motion of a charged particle in the Earth's magnetotail [17].

The outline of the paper is as follows. In Section 2 we formulate the main problem and present the results. Section 3 contains necessary canonical transformations in the system and describes the motion in adiabatic and improved adiabatic approximations. In section 4, we derive the main formulae and estimate error terms. In Concluding remarks we discuss possible perspectives of applying the obtained results in future studies. For clarity of presentation, we restrict ourselves in sections 3 and 4 to the case when coordinates of the saddle and direction of its main axes do not depend on the slow variables [however, in section 2, the result is formulated for the general case].

Some comments on the general case are given in Appendix.

2. Formulation of the results

In this section, we formulate the results of the paper. Detailed description of concepts and notations used here can be found in the following sections.

Consider a dynamical system with Hamiltonian $H = H(p, q, \hat{y}, \hat{x})$, where $q, \varepsilon^{-1}\hat{x}$ are coordinates, and $p, \hat{y}$ are canonically conjugated momenta, $\varepsilon > 0$ is a small parameter, $H \in C^\infty$. The corresponding Hamiltonian equations of motion are:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{\hat{y}} = -\varepsilon\frac{\partial H}{\partial \hat{x}}, \quad \dot{\hat{x}} = \varepsilon\frac{\partial H}{\partial \hat{y}}. \tag{2.1}$$

Variables p, q are called fast variables and correspond to one degree of freedom; variables $\hat{y}, \hat{x}$, corresponding to the other degrees of freedom are called slow variables. Hamiltonian system for p, q at $(\hat{y}, \hat{x}) = \text{const}$ is called fast (or unperturbed) system. The complete system dynamics is described with the use of adiabatic and improved adiabatic approximations. Far from separatrices, action I of the fast system is an adiabatic invariant of the complete (perturbed) system; its value along a trajectory is preserved with the accuracy of order ε on time intervals of order ε^{-1} (see, e.g. [1]). One can introduce improved adiabatic invariant J, which under the same conditions is preserved with the accuracy of order ε^2.

We assume that at all considered values of the slow variables there exists a non-degenerate saddle point C and separatrices l_1, l_2 on the phase plane of the fast system (see figure 1). These separatrices divide the (p, q)-plane into regions $G_i = G_i(\hat{y}, \hat{x})$, $i = 1, 2, 3$. Denote the value of H at the saddle point as $h_c = h_c(\hat{y}, \hat{x})$, and introduce $E = E(p, q, \hat{y}, \hat{x}) = H - h_c$. On the separatrices $E = 0$. We suppose that $E > 0$ in G_3 and $E < 0$ in G_1 and G_2. Areas of regions G_1, G_2 are S_1, S_2 correspondingly, and $S_3 = S_1 + S_2$; $S_i = S_i(\hat{y}, \hat{x})$. Put $\Theta_i = \Theta_i(\hat{y}, \hat{x}) = \{S_i, h_c\}$, where $\{\cdot, \cdot\}$ is the Poisson bracket with respect to variables $(\hat{y}, \hat{x})$: $\{f, g\} = f'_{\hat{x}}g'_{\hat{y}} - f'_{\hat{y}}g'_{\hat{x}}$. We will assume that a phase point in the perturbed motion approaches the separatrix at such values of $\hat{y}, \hat{x}$ that $\Theta_i(\hat{y}, \hat{x})$ are separated from zero: $|\Theta_i(\hat{y}, \hat{x})| > c_1^{-1}$, $c_1 = \text{const} > 0$.

Introduce slow time $\tau = \varepsilon t$. Choose initial conditions of a phase trajectory at $\tau = 0$ in one of the regions G_i far enough from the separatrix, so that the initial value of E is of order 1. Assume that as the slow variables change with time, value of E along the phase trajectory changes and finally the trajectory crosses the separatrix of the fast system either at $\tau > 0$ or at $\tau < 0$ or both at $\tau > 0$ and at $\tau < 0$. Separatrix crossings at $\tau > 0$ and at $\tau < 0$ are characterized by pseudo-phases $\xi, \bar{\xi}$ respectively. We describe below the definition of ξ. Pseudo-phase $\bar{\xi}$ is defined similarly for the motion backwards in time.

When approaching the separatrix, the phase trajectory on the (p, q)-plane near C repeatedly crosses the vertex lying in G_i and bisecting the angle between incoming and outgoing whiskers of C. If $i = 1, 2$, this is either positive or negative part of η-axis, and if $i = 3$ we choose positive direction of ζ-axis (see figure 1). Let E_0 be the value of E

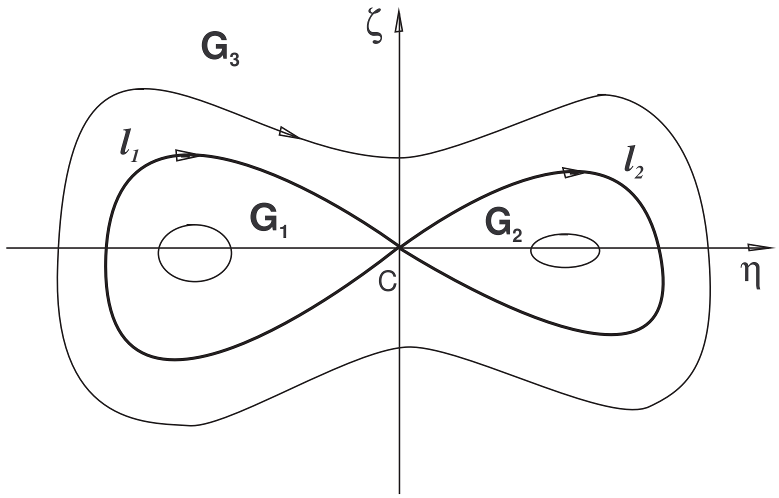

Figure 1. Phase portrait of the fast system.

at the moment when the phase trajectory crosses this vertex for the last time before leaving G_i. Then $\xi = |E_0/(\varepsilon\Theta_{i*})|$, where Θ_{i*} is Θ_i calculated at the moment $\tau = \tau_* > 0$ of separatrix crossing in improved adiabatic approximation (see details in section 3.2). If $\xi > c_2\sqrt{\varepsilon}$, where c_2 is a large enough positive constant, then $c_2\sqrt{\varepsilon} < \xi < 1 + \mathrm{O}(\sqrt{\varepsilon})$, because in this case value E changes by $(-\varepsilon\Theta_{i*} + \mathrm{O}(\varepsilon^{3/2}))$ between two subsequent vertex crossings in G_i [15]. We will assume that $c_2\sqrt{\varepsilon} < \xi < 1 - c_2\sqrt{\varepsilon}$. This assumption means that we do not consider phase points that pass too close to the saddle point C; the measure of correspondingly excluded set of initial conditions is small, of order $\sqrt{\varepsilon}$.

Let I, φ be the action-angle variables of the fast system. As the origin of the angle variable φ we take a point where the phase trajectory crosses the vertex defined in the previous paragraph. Let φ_- be the initial value of the angle variable φ. In the present paper we prove the following formula:

$$\xi = \left\{ \frac{1}{2\pi}\left(\varphi_- + \frac{1}{\varepsilon}\int_0^{\tau_*}(\omega_0(J_-, Y, X) + \varepsilon\omega_1(J_-, Y, X))\mathrm{d}\tau\right) + A_* + \mathrm{O}(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3})\right\}, \tag{2.2}$$

where J_- is the initial value of the improved adiabatic invariant, $X = X(\tau), Y = Y(\tau)$ are slow variables of the improved adiabatic approximation, ω_0 is the frequency in the fast system, ω_1 is the first correction to the frequency. $A_* = A(X(\tau_*), Y(\tau_*))$, $A = 0$ for $i = 1, 2$ and $A = (1/4)(\Theta_2 - \Theta_1)/\Theta_3$ for $i = 3$. Parentheses in (2.2) denote the fractional part.

Value $\bar{\xi}$ is defined exactly like ξ for motion backwards in time. Let $\bar{\tau}_* < 0$ be the moment of separatrix crossing in improved adiabatic approximation. Then from (2.2), by time reversal, we get

$$\bar{\xi} = \left\{ \frac{1}{2\pi}\left(-\varphi_- + \frac{1}{\varepsilon}\int_{\bar{\tau}_*}^{0}(\omega_0(J_-, Y, X) + \varepsilon\omega_1(J_-, Y, X))\mathrm{d}\tau\right) - \bar{A}_* + \mathrm{O}(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3})\right\}, \tag{2.3}$$

where $\bar{A}_* = A(X(\bar{\tau}_*), Y(\bar{\tau}_*))$.

If the trajectory crosses the separatrix both at $\tau > 0$ and $\tau < 0$, we can define both pseudo-phases $\xi, \bar{\xi}$ and obtain the following relation:

$$\xi + \bar{\xi} = \frac{1}{2\pi\varepsilon}\int_{\bar{\tau}_*}^{\tau_*}(\omega_0(J_0, Y, X) + \varepsilon\omega_1(J_0, Y, X))\mathrm{d}\tau + A_* - \bar{A}_* + \mathrm{O}(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}) \bmod 1. \quad (2.4)$$

In the case of explicit slow time dependence formulae (2.2)-(2.4) up to the estimate of the error term give the corresponding formulae in [8].

3. Changes of variables. Adiabatic and improved adiabatic approximations

Assume for simplicity that at all values of the slow variables $\hat{y}, \hat{x}$ point C is at the origin on the phase plane (p, q) of the fast system, and that coordinate axis Cp coincides with the vertex defined in section 2 and is the origin of the phase (i.e. it is the line $\varphi = 0 \bmod 2\pi$). If this is not the case, the result is the same (see Appendix).

3.1. Canonical variables

In the fast system, action-angle variables I, $\varphi \bmod 2\pi$ are introduced by a canonical transformation of variables defined by generating function $W(I, q, \hat{y}, \hat{x})$ containing $\hat{y}, \hat{x}$ as parameters. We take this function in the form

$$W(I, q, \hat{y}, \hat{x}) = \int_0^q \mathcal{P}(I, q', \hat{y}, \hat{x})\mathrm{d}q', \quad (3.1)$$

where $\mathcal{P}$ is the value of p-variable along the trajectory with the prescribed value of action I. In the new variables the Hamiltonian has the form $H = H_0(I, \hat{y}, \hat{x})$. Now, in the complete system, make a canonical transformation of variables $(p, q, \hat{y}, \hat{x}) \mapsto (\bar{I}, \bar{\varphi}, \bar{y}, \bar{x})$ with the generating function $\bar{y}\varepsilon^{-1}\hat{x} + W(\bar{I}, q, \bar{y}, \hat{x})$. The canonically conjugated pairs of variables are $(\bar{I}, \bar{\varphi})$ and $(\bar{y}, \varepsilon^{-1}\bar{x})$. Formulae for the transformation of variables are:

$$\bar{\varphi} = \partial W/\partial \bar{I}, \;\; p = \partial W/\partial q, \;\; \bar{x} = \hat{x} + \varepsilon\partial W/\partial \bar{y}, \;\; \hat{y} = \bar{y} + \varepsilon \partial W/\partial \hat{x}. \quad (3.2)$$

In the new variables, Hamiltonian H has the form

$$H = H_0(\bar{I}, \bar{y}, \bar{x}) + \varepsilon H_1(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}) + \varepsilon^2 H_2(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}, \varepsilon), \quad (3.3)$$

where

$$H_1 = \frac{\partial H}{\partial \hat{y}}\frac{\partial W}{\partial \hat{x}} - \frac{\partial H_0}{\partial \hat{x}}\frac{\partial W}{\partial \bar{y}};$$

$$H_2 = \left[\frac{1}{2}\frac{\partial^2 H}{\partial \hat{y}^2}\left(\frac{\partial W}{\partial \hat{x}}\right)^2 + \frac{1}{2}\frac{\partial^2 H_0}{\partial \hat{x}^2}\left(\frac{\partial W}{\partial \bar{y}}\right)^2 + \frac{\partial H_0}{\partial \hat{x}}\frac{\partial^2 W}{\partial \hat{x}\partial \hat{y}}\frac{\partial W}{\partial \hat{y}} - \frac{\partial^2 H}{\partial \hat{x}\partial \hat{y}}\frac{\partial W}{\partial \hat{x}}\frac{\partial W}{\partial \hat{y}} - \frac{\partial H}{\partial \hat{y}}\frac{\partial^2 W}{\partial \hat{x}^2}\frac{\partial W}{\partial \hat{y}}\right]_{im} \quad (3.4)$$

and function H_2 is 2π-periodic in $\bar{\varphi}$. The above expression for H_2 was obtained using the mean value theorem of calculus several times; subscript "*im*" denotes that partial derivatives of H, H_0, W in this expression are calculated in appropriate intermediate points whose positions depend on ε. In what follows, specific positions of these points are not essential. In the adiabatic approximation, dynamics is described by Hamiltonian H_0. In this approximation $\bar{I} = \text{const}$ along a phase trajectory.

Now we construct a canonical, close to identical, transformation of variables $(\bar{I}, \bar{\varphi}, \bar{y}, \bar{x}) \mapsto (J, \psi, y, x)$ in order to make the terms of order ε in the Hamiltonian independent of phase. We look for the generating function of the transformation in the form $J\bar{\varphi} + y\varepsilon^{-1}\bar{x} + \varepsilon W_1(J, \bar{\varphi}, y, \bar{x})$. We have:

$$\bar{I} = J + \varepsilon \partial W_1/\partial\bar{\varphi},\ \psi = \bar{\varphi} + \varepsilon \partial W_1/\partial J,\ \bar{y} = y + \varepsilon^2 \partial W_1/\partial \bar{x},\ x = \bar{x} + \varepsilon^2 \partial W_1/\partial y. \tag{3.5}$$

In the new variables, the Hamiltonian takes the form:

$$\mathcal{H} = H_0(J, y, x) + \varepsilon \mathcal{H}_1(J, y, x) + \varepsilon^2 \mathcal{H}_2(J, \psi, y, x, \varepsilon). \tag{3.6}$$

Substitute expressions (3.5) into equation $H = \mathcal{H}$, where H is given by (3.3), (3.4) and equate terms of similar order in ε. Then in the first order we have:

$$\frac{\partial H_0}{\partial J}\frac{\partial W_1}{\partial \bar{\varphi}} + H_1(J, \bar{\varphi}, y, x) = \mathcal{H}_1(J, y, x). \tag{3.7}$$

Averaging over $\bar{\varphi}$, we obtain

$$\mathcal{H}_1 = \langle H_1 \rangle, \tag{3.8}$$

where the brackets denote averaging. For W_1 we find

$$\frac{\partial W_1}{\partial \bar{\varphi}} = -\frac{1}{\partial H_0/\partial J}\left(H_1 - \langle H_1 \rangle\right). \tag{3.9}$$

Hence,

$$W_1 = -\frac{1}{\partial H_0/\partial J}\int_0^{\bar{\varphi}} \left(H_1 - \langle H_1 \rangle\right) \mathrm{d}\varphi + C(J, y, \bar{x}), \tag{3.10}$$

where $C(J, y, \bar{x})$ is an arbitrary function. In what follows, we choose $C(J, y, \bar{x}) \equiv 0$. Equating terms of higher order in ε, we obtain the following expression for $\mathcal{H}_2$:

$$\mathcal{H}_2 = \left[\frac{1}{2}\frac{\partial^2 H_0}{\partial J^2}\left(\frac{\partial W_1}{\partial \bar{\varphi}}\right)^2 + \frac{\partial H_1}{\partial J}\frac{\partial W_1}{\partial \bar{\varphi}} + \frac{\partial H_0}{\partial \bar{y}}\frac{\partial W_1}{\partial \bar{x}} - \frac{\partial H_0}{\partial \bar{x}}\frac{\partial W_1}{\partial y}\right]_{im} + H_2, \tag{3.11}$$

where H_2 is given by (3.4) and, like in (3.4), partial derivatives are calculated in appropriate intermediate points to make the equality exact.

In improved adiabatic approximation, the dynamics is described by the Hamiltonian $H_0(J, Y, X) + \varepsilon \mathcal{H}_1(J, Y, X)$. In this approximation J is an integral of motion. With the accuracy of order ε^2, the following formula for J is valid (see [15]):

$$J = J(p, q, \hat{y}, \hat{x}) = I + \varepsilon u(p, q, \hat{y}, \hat{x}), \tag{3.12}$$

$$\begin{aligned} u = \frac{1}{4\pi}&\left[\int_0^T \left(\frac{\partial E}{\partial \hat{y}}\int_0^t \frac{\partial E}{\partial \hat{x}}\mathrm{d}\sigma\right)\mathrm{d}t - \int_0^T \left(\frac{\partial E}{\partial \hat{x}}\int_0^t \frac{\partial E}{\partial \hat{y}}\mathrm{d}\sigma\right)\mathrm{d}t\right] \\ &+ \frac{1}{2\pi}\int_0^T \left(\frac{T}{2} - t\right)\{E, h_c\}\,\mathrm{d}t. \end{aligned} \tag{3.13}$$

The integrals here are calculated along a phase trajectory of the fast system passing through the point (p, q); σ is the time of motion along this trajectory starting from this point, T is the period of motion. Function J is the improved adiabatic invariant. In the complete system far from separatrices this value is constant with the accuracy of order ε^2 on time intervals of order ε^{-1}.

3.2. Separatrix crossing in the adiabatic approximation

Value $\varepsilon\Theta_i(\hat{y}, \hat{x})$ defined in section 2 is the rate of change of S_i in the adiabatic approximation as a phase point in G_i approaches the separatrix. According to our assumption, $\Theta_i(\hat{y}, \hat{x}) > \text{const} > 0$ in the considered region of variation of the slow variables. The action $I(H, \hat{y}, \hat{x})$ is the area encircled by a phase trajectory, divided by 2π, and in the adiabatic approximation $I = \text{const}$. Therefore, in this approximation phase points initially in G_3 can cross the separatrix in a finite interval of slow time εt, and then enter G_1 or G_2.

In our analysis we have to consider the separatrix crossing in the improved adiabatic approximation. We use the following scheme to describe variation of J, ψ and the slow variables. Let the motion start at $t = 0$ at a point $M_-(p_-, q_-, \hat{y}_-, \hat{x}_-)$, and $(p_-, q_-) \in G_3(\hat{y}_-, \hat{x}_-)$. Let $J = J_-$ at this point. Evolution of slow variables is described by solution $Y(\tau), X(\tau)$, $\tau = \varepsilon t$ of the slow system with Hamiltonian $H_0(J_-, Y, X) + \varepsilon\mathcal{H}_1(J_-, Y, X)$ with initial conditions $Y(0) = \bar{y}_-, X(0) = \bar{x}_-$. The relations between $\bar{y}_-, \bar{x}_-$ here and $\hat{y}_-, \hat{x}_-$ are given by (3.2). The time moment of the separatrix crossing τ_* in this approximation can be found from the equation $S_3(Y(\tau_*), X(\tau_*)) = 2\pi J_-$.

Introduce function $\bar{E} = \bar{E}(J, Y, X)$ with the following equation:

$$I(\bar{E} + h_c(Y, X), Y, X) = J. \tag{3.14}$$

Along a trajectory of the improved adiabatic approximation $Y = Y(\tau)$, $X = X(\tau)$ and $J = J_-$. Therefore, in this case from (3.14) $\bar{E} = \bar{E}(J_-, Y(\tau), X(\tau))$ and $\bar{E}$ is the energy reserve on the way to the separatrix in this approximation. In particular, at $\tau = \tau_*$ we have $\bar{E} = 0$.

4. Formula for the pseudo-phase

For definiteness, we consider motion in domain G_3. The result for domains G_1, G_2 is the same. For simplicity of presentation we assume that during the motion $0 < E < c_3^{-1} < 1/2$, where c_3 is a big enough constant. The case when there is a segment of trajectory with $E > c_3^{-1}$ does not create difficulties, because the improved adiabatic approximation allows to describe behavior of phase φ on such a segment with the accuracy $\mathrm{O}(\varepsilon)$, which is good enough for result (2.2).

We consider the motion on a certain slow time interval $0 < \tau < \tau_k$. Here τ_k is a slow time moment close enough to τ_*, $\tau_k < \tau_*$. We choose τ_k in such a way that at $0 \le \tau \le \tau_k$ one has $\bar{E} \gtrsim \varepsilon^{2/3}|\ln\varepsilon|^{2/3}$ along the trajectory of the adiabatic approximation. This choice

τ_k will be justified later. According to [18], at $0 \le \tau \le \tau_*$ one has $|E - \bar{E}| < O(\varepsilon)$. Hence, if $\bar{E} > c_4\varepsilon$ with large enough constant c_4, then

$$\frac{1}{2}\bar{E} \le E \le 2\bar{E}. \tag{4.1}$$

This relation implies that in estimates of the kind $\mathrm{O}(E^{\beta_1}|\ln E|^{\beta_2})$ one can replace E with $\bar{E}$ and vice versa.

We start with a formula for variation of ψ with time. Let $\psi = \psi_-$ at time moment $\tau = 0$. From the Hamiltonian equations of motion we have:

$$\begin{aligned}\psi(\tau) = \psi_- + \varepsilon^{-1}\int_0^\tau &[\omega_0(J, y(\tau'), x(\tau')) + \varepsilon\omega_1(J, y(\tau'), x(\tau')) \\ &+\varepsilon^2\omega_2(J, \psi, y(\tau'), x(\tau'))]\,\mathrm{d}\tau',\end{aligned} \tag{4.2}$$

where

$$\omega_0 = \partial H_0/\partial J, \quad \omega_1 = \partial\mathcal{H}_1/\partial J, \quad \omega_2 = \partial\mathcal{H}_2/\partial J,$$

and (J, ψ), $(y, \varepsilon^{-1}x)$ are canonically conjugated pairs of variables defined in (3.5). Rewrite (4.2) as follows:

$$\psi(\tau) = \psi_- + \varepsilon^{-1}\int_0^\tau [\omega_0(J_-, Y(\tau'), X(\tau')) + \varepsilon\omega_1(J_-, Y(\tau'), X(\tau'))]\,\mathrm{d}\tau' + D, \tag{4.3}$$

where we introduced notation

$$\begin{aligned}D = \varepsilon^{-1}\int_0^\tau \Bigg[&\left(\frac{\partial\omega_0}{\partial J} + \varepsilon\frac{\partial\omega_1}{\partial J}\right)(J - J_-) + \left(\frac{\partial\omega_0}{\partial Y} + \varepsilon\frac{\partial\omega_1}{\partial Y}\right)(Y - y) \\ &+ \left(\frac{\partial\omega_0}{\partial X} + \varepsilon\frac{\partial\omega_1}{\partial X}\right)(X - x) + \varepsilon^2\omega_2(J, \psi, y, x)\Bigg]\,\mathrm{d}\tau',\end{aligned} \tag{4.4}$$

and Y, X in (4.3), (4.4) are slow variables of the improved adiabatic approximation. Partial derivatives in (4.4) are calculated at proper intermediate points to make the equality in (4.3) exact.

Our primary aim is to estimate D. Using (3.14), one can represent terms under integral in (4.3) as functions of $(\bar{E}(J, Y, X), Y, X)$. In the main approximation we have

$$\omega_0 = \partial\bar{E}/\partial J \sim |\ln\bar{E}|^{-1}. \tag{4.5}$$

Symbols "$\sim$" and "$\mathrm{O}(\cdot)$" here and below in the paper imply that relations containing them can be differentiated: if two functions are related via such a symbol, then their derivatives are also related via the same symbol. To ensure this property, we control singular terms in our estimates.

Therefore,

$$\partial\omega_0/\partial J = \partial\omega_0/\partial\bar{E}\cdot\partial\bar{E}/\partial J \sim \bar{E}^{-1}(\ln|\bar{E}|)^{-3}.$$

To estimate $\partial\omega_1/\partial J$, we need first to estimate $\mathcal{H}_1$. We do this with the use of (3.4). We have (see [15]):

$$\frac{\partial W}{\partial\alpha} = \int_0^t \left(\left\langle\frac{\partial E}{\partial\alpha}\right\rangle - \frac{\partial E}{\partial\alpha}\right)\mathrm{d}t', \quad \alpha = \hat{y}, \hat{x}, \tag{4.6}$$

where the integral is taken along a phase trajectory of the fast system, and t is the time of motion along this trajectory to the point where $\partial W/\partial\alpha$ is calculated. Using the following formula (see [15]):

$$\oint_{E=h} \frac{\partial E}{\partial \alpha}\,\mathrm{d}t = -\frac{\partial S_i}{\partial \alpha} + \mathrm{O}(h\ln|h|), \quad \alpha = \hat{y}, \hat{x}, \tag{4.7}$$

we find that $\partial W/\partial\alpha = \mathrm{O}(1) + \mathrm{O}((\ln\bar{E})^{-1})$ and hence, from (3.4),(3.8),

$$\mathcal{H}_1 = \mathrm{O}(1) + \mathrm{O}((\ln\bar{E})^{-1}). \tag{4.8}$$

We have to retain the second term, because it can be essential after differentiating. Thus, using (4.8) and (4.5,

$$\omega_1 = \mathrm{O}(\bar{E}^{-1}(\ln\bar{E})^{-3}), \quad \partial\omega_1/\partial J = \mathrm{O}(\bar{E}^{-2}(\ln\bar{E})^{-4}).$$

To estimate derivatives like $\partial\bar{E}/\partial Y$, one can differentiate equality (3.14) to obtain $\partial\bar{E}/\partial Y \sim (\ln\bar{E})^{-1}$. Then we find:

$$\partial\omega_0/\partial Y \sim \bar{E}^{-1}(\ln\bar{E})^{-3}, \;\; \partial\omega_1/\partial Y = \mathrm{O}(\bar{E}^{-2}(\ln\bar{E})^{-4}),$$

and similarly for X.

From expression (3.11) for $\mathcal{H}_2$ using the above estimates, and the estimate

$$\frac{\partial W_1}{\partial \varphi} = -u = \mathrm{O}(\ln\bar{E}) \tag{4.9}$$

(see [15]), we find

$$\mathcal{H}_2 = \mathrm{O}(\bar{E}^{-1}(\ln\bar{E})^{-1}). \tag{4.10}$$

Hence,

$$\omega_2 = \mathrm{O}(\bar{E}^{-2}(\ln\bar{E})^{-2}).$$

Estimate the factors $(X-x)$ and $(Y-y)$ in (4.4). From the Hamiltonian equations of motion we have

$$\begin{aligned}\dot{X} &= \varepsilon\frac{\partial H_0(J_-,Y,X)}{\partial Y} + \varepsilon^2\frac{\partial \mathcal{H}_1(J_-,Y,X)}{\partial Y},\\ \dot{x} &= \varepsilon\frac{\partial H_0(J,y,x)}{\partial y} + \varepsilon^2\frac{\partial \mathcal{H}_1(J,y,x)}{\partial y} + \varepsilon^2\frac{\partial \mathcal{H}_2(J,y,x)}{\partial y},\end{aligned}$$

where the partial derivative of $\mathcal{H}_2$ in the second equation is calculated at an appropriate point to make the equation exact, and analogously for $\dot{Y}, \dot{y}$. Hence, the value $\Delta(\tau) = |X-x| + |Y-y|$ at the moment when $\Delta \neq 0$ satisfies differential inequality:

$$\Delta' \leq \varepsilon a(\tau)\Delta + \varepsilon^3 b(\tau) \tag{4.11}$$

with initial condition $\Delta(0) = \mathrm{O}(\varepsilon^2)$, where $a(\tau) > 0,\ b(\tau) > 0$,

$$\begin{aligned}a(\tau) &= \mathrm{O}\left(\left|\frac{\partial^2 H_0}{\partial X^2}\right| + \left|\frac{\partial^2 H_0}{\partial Y^2}\right| + \left|\frac{\partial^2 H_0}{\partial X\partial Y}\right|\right) + \varepsilon\mathrm{O}\left(\left|\frac{\partial^2 \mathcal{H}_1}{\partial X^2}\right| + \left|\frac{\partial^2 \mathcal{H}_1}{\partial Y^2}\right| + \left|\frac{\partial^2 \mathcal{H}_1}{\partial X\partial Y}\right|\right)\\ &= \mathrm{O}\left(\frac{1}{\bar{E}(\ln\bar{E})^3}\right) + \varepsilon\mathrm{O}\left(\frac{1}{\bar{E}^2(\ln\bar{E})^4}\right) = \mathrm{O}\left(\frac{1}{\bar{E}(\ln\bar{E})^3}\right)\end{aligned}$$

$$
\begin{aligned}
b(\tau) = {} & \mathrm{O}\left(\left|\frac{\partial \mathcal{H}_2}{\partial X}\right| + \left|\frac{\partial \mathcal{H}_2}{\partial Y}\right|\right) + \varepsilon^{-2}\mathrm{O}\left(\left|\frac{\partial^2 H_0}{\partial Y \partial J}\right| + \left|\frac{\partial^2 H_0}{\partial X \partial J}\right|\right)|J - J_-| \\
& + \varepsilon^{-1}\mathrm{O}\left(\left|\frac{\partial^2 \mathcal{H}_1}{\partial Y \partial J}\right| + \left|\frac{\partial^2 \mathcal{H}_1}{\partial X \partial J}\right|\right)|J - J_-| \\
= {} & \mathrm{O}\left(\frac{1}{\bar{E}^2(\ln\bar{E})^2}\right) + \varepsilon^{-2}|J - J_-|\left[\mathrm{O}\left(\frac{1}{\bar{E}(\ln\bar{E})^3}\right) + \varepsilon\mathrm{O}\left(\frac{1}{\bar{E}^2(\ln\bar{E})^4}\right)\right] \\
= {} & \mathrm{O}\left(\frac{1}{\bar{E}^2(\ln\bar{E})^2}\right) + \mathrm{O}\left(\frac{1}{\bar{E}^2(\ln\bar{E})^3}\right) = \mathrm{O}\left(\frac{1}{\bar{E}^2(\ln\bar{E})^2}\right).
\end{aligned} \tag{4.12}
$$

Here we have used estimates (4.8), (4.10), (4.1) and the estimate $|J - J_-| = \mathrm{O}(\varepsilon^2/\bar{E})$ (see [15]). Solving (4.11), we get

$$
\Delta(\tau) \leq \exp\left[\int_0^\tau a(\tau')\,\mathrm{d}\tau'\right]\left(\Delta(0) + \varepsilon^2\int_0^\tau b(\tau')\,\mathrm{d}\tau'\right). \tag{4.13}
$$

To calculate the integrals in this expression, we make the change of variable $\mathrm{d}\tau = (\mathrm{d}\tau/\mathrm{d}\bar{E})\mathrm{d}\bar{E}$ and use equality

$$
\frac{\mathrm{d}\tau}{\mathrm{d}\bar{E}} = \frac{1}{\mathrm{d}\bar{E}/\mathrm{d}\tau} = -T/(\Theta + \mathrm{O}(\bar{E}\ln\bar{E})) \sim \ln\bar{E}. \tag{4.14}
$$

Straightforward calculation using the above estimates for a and b gives the following estimate:

$$
\Delta(\tau) = \mathrm{O}\left(\frac{\varepsilon^2}{\bar{E}\ln\bar{E}}\right). \tag{4.15}
$$

Substituting all the above estimates into (4.4) and changing the variable of integration from τ to $\bar{E}$ like in (4.14), we obtain the estimate:

$$
D = \mathrm{O}(\varepsilon\bar{E}^{-1}(\ln\bar{E})^{-1}). \tag{4.16}
$$

Now let $\tau = \tau_k$, where τ_k is a slow time moment when the trajectory crosses the axis $C\zeta$ bisecting the angle between invariant manifolds of C (see figure 1). According to our assumptions, at this time moment $q = 0$, $\varphi = 0 \bmod 2\pi$ and, as it follows from (3.1), (3.2), (3.5), (3.10), $\psi = \bar{\varphi} = \varphi$.

We assume that at this moment the phase point is still far enough from the separatrix, and it still has to cross $C\zeta$ a number of times before crossing the separatrix. From (4.3), (4.16), we conclude that

$$
2\pi N = \varphi_- + \varepsilon^{-1}\int_0^{\tau_*}(\omega_0 + \varepsilon\omega_1)\,\mathrm{d}\tau - \varepsilon^{-1}\int_{\tau_k}^{\tau_*}(\omega_0 + \varepsilon\omega_1)\,\mathrm{d}\tau + \mathrm{O}(\varepsilon\bar{E}_k^{-1}(\ln\bar{E}_k)^{-1}), \tag{4.17}
$$

where N is an integer, $\bar{E}_k$ is value of $\bar{E}$ at $\tau = \tau_k$, and ω_0, ω_1 are functions of $(J_-, Y(\tau), X(\tau))$. To calculate integral

$$
\int_{\tau_k}^{\tau_*}\omega_0\,\mathrm{d}\tau
$$

we make the change of variable $\mathrm{d}\tau = (\mathrm{d}\tau/\mathrm{d}\bar{E})\mathrm{d}\bar{E}$ as above. At $\tau = \tau_*$, we have $\bar{E} = 0$. Making use of (4.14), we find:

$$
\int_{\tau_k}^{\tau_*}\omega_0\,\mathrm{d}\tau = \frac{2\pi}{\Theta_*}\bar{E}_k + \mathrm{O}(\bar{E}_k^2\ln\bar{E}_k). \tag{4.18}
$$

The error term in (4.18) takes into account both error term in (4.14) and variation of Θ with time.

Our next task is to find the relation between $\bar{E}$ and E. Using definition of $\bar{E}$ (3.14) we obtain:

$$H_0(J_-, Y, X) = \bar{E} + h_c(Y, X). \tag{4.19}$$

Using (3.6) and (4.10) one can write

$$\bar{E} = H - h_c(Y, X) - \varepsilon \mathcal{H}_1(J_-, Y, X) + \mathrm{O}(\varepsilon^2). \tag{4.20}$$

According to (4.15), $|x - X| + |y - Y| = \mathrm{O}(\varepsilon^2 \bar{E}^{-1} (\ln \bar{E})^{-1})$. Therefore,

$$\bar{E} = H - h_c(y, x) - \varepsilon \mathcal{H}_1(J_-, Y, X) + \mathrm{O}(\varepsilon^2 \bar{E}^{-1} (\ln \bar{E})^{-1}). \tag{4.21}$$

Consider this relation at the time moment $\tau = \tau_k$. At this moment $q = 0$. Formulae for generating functions W, W_1 (3.1), (3.10) and for transformations of variables (3.2), (3.5) imply that at this moment values of variables y, x and original variables $\hat{y}, \hat{x}$ coincide. Thus we have

$$\bar{E}_k = E_k - \varepsilon \mathcal{H}_1(J_-, Y, X)_k + \mathrm{O}(\varepsilon^2\, \bar{E}_k^{-1} (\ln \bar{E}_k)^{-1})$$

and

$$\int_{\tau_k}^{\tau_*} \omega_0 \, \mathrm{d}\tau = \frac{2\pi}{\Theta_*}(E_k - \varepsilon(\mathcal{H}_1)_k) + \mathrm{O}(E_k^2 \ln E_k) + \mathrm{O}(\varepsilon^2\, E_k^{-1} (\ln E_k)^{-1}), \tag{4.22}$$

where E_k, $(\mathcal{H}_1)_k$ are values of E and $\mathcal{H}_1$ accordingly at the point corresponding to the slow time moment τ_k.

Calculate now the integral

$$\varepsilon \int_{\tau_k}^{\tau_*} \omega_1 \, \mathrm{d}\tau.$$

We change the integration variable like in (4.18) to obtain

$$\varepsilon \int_{\tau_k}^{\tau_*} \omega_1 \, \mathrm{d}\tau = \varepsilon \int_0^{\bar{E}_k} \frac{\partial \mathcal{H}_1}{\partial \bar{E}} \frac{2\pi}{\Theta} (1 + \mathrm{O}(\bar{E} \ln \bar{E}))\, \mathrm{d}\bar{E}. \tag{4.23}$$

We have:

$$\frac{\partial \mathcal{H}_1}{\partial \bar{E}} = \frac{\mathrm{d}\mathcal{H}_1}{\mathrm{d}\bar{E}} - \frac{\partial \mathcal{H}_1}{\partial X} \cdot \frac{\mathrm{d}X}{\mathrm{d}\bar{E}} - \frac{\partial \mathcal{H}_1}{\partial Y} \cdot \frac{\mathrm{d}Y}{\mathrm{d}\bar{E}} \ ,$$

where the X, Y-derivatives of $\mathcal{H}_1$ are calculated at constant $\bar{E}$. This expression should be evaluated along the trajectory of the improved adiabatic approximation. For the derivatives of X and Y, one can write $\dot{X} = \varepsilon (\mathrm{d}X/\mathrm{d}\bar{E})(\mathrm{d}\bar{E}/\mathrm{d}\tau)$ and then use the Hamiltonian equations of the improved adiabatic approximation to find expressions for $\dot{X}, \dot{Y}$. Thus, we obtain

$$\frac{\partial \mathcal{H}_1}{\partial \bar{E}} - \frac{\mathrm{d}\mathcal{H}_1}{\mathrm{d}\bar{E}} = \frac{1}{\mathrm{d}\bar{E}/\mathrm{d}\tau} \left[- \left(\frac{\partial \mathcal{H}_1}{\partial X} \right)_{\bar{E}} \cdot \frac{\partial H_0}{\partial Y} + \left(\frac{\partial \mathcal{H}_1}{\partial Y} \right)_{\bar{E}} \cdot \frac{\partial H_0}{\partial X} \right].$$

The terms in square brackets are $\mathrm{O}(1)$, and $\mathrm{d}\bar{E}/\mathrm{d}\tau \sim (\ln \bar{E})^{-1}$. Hence,

$$\frac{\partial \mathcal{H}_1}{\partial \bar{E}} = \frac{\mathrm{d}\mathcal{H}_1}{\mathrm{d}\bar{E}} + \mathrm{O}(\ln \bar{E}).$$

Substituting this expression into (4.23) one obtains:

$$\varepsilon \int_{\tau_k}^{\tau_*} \omega_1 \,\mathrm{d}\tau = \varepsilon \frac{2\pi}{\Theta_*} \left[(\mathcal{H}_1)_k - (\mathcal{H}_1)_*\right] + \varepsilon \mathrm{O}(\bar{E}_k \ln \bar{E}_k), \tag{4.24}$$

where $(\mathcal{H}_1)_*$ is a value of $\mathcal{H}_1$ at $\tau = \tau_*$. Combining (4.22) and (4.24), we get:

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\tau_k}^{\tau_*} (\omega_0 + \varepsilon\omega_1)\mathrm{d}\tau &= \frac{2\pi}{\varepsilon\Theta_*} E_k - \frac{2\pi}{\Theta_*}(\mathcal{H}_1)_* \\ &+ \mathrm{O}(\varepsilon^{-1} E_k^2 \ln E_k) + \mathrm{O}(E_k \ln E_k) + \mathrm{O}(\varepsilon\, E_k^{-1} (\ln E_k)^{-1}).\end{aligned} \tag{4.25}$$

After slow time moment τ_k the phase trajectory approaches the separatrix, repeatedly crossing the axis $C\zeta$. Values of E at these crossings form a decreasing sequence $\{E_n\}$. Let $n = 0$ at the last crossing of $C\zeta$ before crossing the separatrix, and let value of n grow with time [i.e. n takes negative values before the separatrix crossing]. In order to find the difference $E_n - E_{n-1}$, one should integrate $\mathrm{d}E/\mathrm{d}t$ along one turn of the phase trajectory in the exact system. According to [3], [15],

$$E_n - E_{n-1} = \varepsilon \oint_{E=E_n} \{E, h_c\} \mathrm{d}t + \varepsilon^2 \mathrm{O}(E_n^{-1/2}). \tag{4.26}$$

The integral in (4.26) is calculated along the trajectory of the fast system with $E = E_n$ and fixed values $x = x_n, y = y_n$ corresponding to the time moment of the n-th crossing of $C\zeta$. [In the particular case under consideration, when position of the saddle point C does not depend on slow variables $\hat{y}, \hat{x}$, the estimate of the error term in (4.26) can be replaced with $\varepsilon^2 \mathrm{O}(\ln E_n)$. However, this improvement of the intermediate estimate does not produce improvement of error estimates in final formulae. We use the error term estimate in the form presented in (4.26), since this estimate is also valid in the general case, when position of C depends on $\hat{y}, \hat{x}$; this case is discussed in Appendix.] We have [15]

$$\begin{aligned}&\oint_{E=E_n} \{E, h_c\} \mathrm{d}t = -\Theta(x_n, y_n) + \mathrm{O}(E_n \ln E_n), \\ &|x_n - x_0| + |y_n - y_0| = \mathrm{O}(E_n \ln E_n).\end{aligned} \tag{4.27}$$

From (4.26), (4.27) we find

$$E_n - E_{n-1} = -\varepsilon\Theta_0 + \varepsilon \mathrm{O}(E_n \ln E_n) + \varepsilon^2 \mathrm{O}(E_n^{-1/2}), \tag{4.28}$$

where Θ_0 is the value of Θ at the time moment of the last crossing of $C\zeta$. Thus, with each turn E decreases by a value approximately equal to $\varepsilon\Theta_0$. Summing up expressions of the kind of (4.28) one obtains:

$$E_n = E_0 + \varepsilon\Theta_0 n + \mathrm{O}(E_n^2 \ln E_n) + \varepsilon \mathrm{O}(\sqrt{E_n}). \tag{4.29}$$

Let $n = m$ at the slow time moment $\tau = \tau_k$. Substitute (4.29) into (4.25). The result is:

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\tau_k}^{\tau_*} (\omega_0 + \varepsilon\omega_1)\mathrm{d}\tau &= \frac{2\pi}{\varepsilon\Theta_*}(E_0 + \varepsilon\Theta_0 m) - \frac{2\pi}{\Theta_*}(\mathcal{H}_1)_* \\ &+ \mathrm{O}(\varepsilon^{-1} E_m^2 \ln E_m) + \mathrm{O}(\sqrt{E_m}) + \mathrm{O}(\varepsilon\, E_m^{-1} (\ln E_m)^{-1}).\end{aligned} \tag{4.30}$$

Now we can substitute (4.30) into (4.17). To minimize the error term we choose $E_m \sim \varepsilon^{2/3}|\ln\varepsilon|^{-2/3}$ (i.e. $m \sim \varepsilon^{-1/3}|\ln\varepsilon|^{-2/3}$). Take into account that $\Theta_0 - \Theta_* = \mathrm{O}(\varepsilon)$. Thus we obtain:

$$2\pi N = \varphi_0 + \frac{1}{\varepsilon}\int_0^{\tau_*}(\omega_0 + \varepsilon\omega_1)\mathrm{d}\tau - 2\pi\xi + \frac{2\pi}{\Theta_*}(\mathcal{H}_1)_* + \mathrm{O}(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}),$$

where N is an integer, and $\xi = E_0/(\varepsilon\Theta_*)$. Hence,

$$2\pi\xi = \varphi_0 + \frac{1}{\varepsilon}\int_0^{\tau_*}(\omega_0 + \varepsilon\omega_1)\mathrm{d}\tau + \frac{2\pi}{\Theta_*}(\mathcal{H}_1)_* + \mathrm{O}(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}) \mod 2\pi. \tag{4.31}$$

Note, that this formula is valid not only in the case when a phase trajectory approaches the separatrix from region G_3, but also for G_1 and G_2.

Consider the term $(\mathcal{H}_1)_*$. We have the following formula valid at the points of (p, q)-plane belonging to the axis where the origin for the angle variable is chosen (see [15]):

$$\mathcal{H}_1 = -\omega_0 u. \tag{4.32}$$

Value of function $u(p, q, \hat{y}, \hat{x})$ is given by formula (3.13).

Let a point (p, q) on the phase plane of the fast system lie in G_i, $i = 1, 2$ near C on the axis $C\eta$ (see figure 1). Then [15]

$$2\pi u = d_i + \mathrm{O}(\sqrt{|h|}\ln|h|),\ h = E(p, q, \hat{y}, \hat{x}), \tag{4.33}$$

where d_i and a, b_i below are smooth functions of $(\hat{y}, \hat{x})$.

If a point (p, q) lies in G_3 near C on the positive part of $C\zeta$, then [15]

$$\begin{aligned} 2\pi u &= (a/2)(\Theta_2 - \Theta_1)\ln h + (1/2)(\Theta_1 b_2 - \Theta_2 b_1) + (1/2)\{S_2, S_1\} + d_3 + \mathrm{O}(\sqrt{h}\ln h), \\ d_3 &= d_1 + d_2. \end{aligned} \tag{4.34}$$

[In the particular case under consideration when position of saddle point C does not depend on values of $\hat{y}, \hat{x}$, error estimates in (4.33), (4.34) can be replaced with $\mathrm{O}(h\ln^2|h|)$.] In the region G_i we have [15] $\omega_0 = 2\pi/T_i$,

$$\begin{aligned} T_i &= -a_i\ln|h| + b_i + \mathrm{O}(h\ln|h|), \\ a_1 &= a_2 = a,\ a_3 = 2a,\ b_3 = b_1 + b_2. \end{aligned} \tag{4.35}$$

Hence, from (4.32), (4.33), (4.35) we conclude that in the regions G_1, G_2 we have $(\mathcal{H}_1)_* = 0$. In G_3, we find from (4.32), (4.34), (4.35) that

$$\frac{2\pi}{\Theta_*}(\mathcal{H}_1)_* = \frac{\pi}{2}\ \cdot\ \frac{\Theta_{2*} - \Theta_{1*}}{\Theta_{3*}}. \tag{4.36}$$

Therefore, we finally obtain formula (2.2).

5. Concluding remarks

The error term estimate in (2.2) is slightly worse than one obtained in [8], that is $\mathrm{O}(\varepsilon^{1/3}|\ln\varepsilon|^{-2/3})$, but is a little better than one presented in [11] ($\mathrm{O}(\varepsilon^{1/3}|\ln\varepsilon|^{1/3})$. The both latter estimates were obtained for the system explicitly depending on the slow time. In fact, the most essential property is that this error term is $o(1)$, i.e. tends to zero as $\varepsilon \to 0$.

Formula (2.2), together with the formula for the change of J at separatrix crossing obtained in [15], can be used to produce a map describing dynamics in the systems of two degrees of freedom with separatrix crossings for a majority of initial conditions. It turns out that this map, up to the error terms, is the same as in the system with explicit slow time dependence. In [11, 12] on the base of this map in the symmetric case $\Theta_1 \equiv \Theta_2$, existence of stable periodic trajectories in the region with separatrix crossings was proved. The results of the present paper and [15] allow to use the theorem of [11, 12] to establish existence of stable periodic trajectories in the slow-fast system under consideration. The number of this trajectories on an energy level is of order ε^{-1}, and on the phase plane of the map each of these trajectories is surrounded by a stability island of the area of order ε. Such islands in systems taken from [16, 17] were recently found numerically in [19, 20]. We plan to describe these topics in more details in a future paper.

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Appendix

In section 4, formula (2.2) was obtained under additional assumption that the saddle point does not move on the phase plane of the fast variables and the directions of the main axes of the saddle do not change with variation of the slow variables. In this Appendix, we demonstrate that formula (2.2) stays valid without this assumption.

Let $p = p_c(\hat{y}, \hat{x}),\ q = q_c(\hat{y}, \hat{x})$ be coordinates of the saddle point C. At frozen values of the slow variables one can make canonical transformation of variables $(p, q) \mapsto (\tilde{p}, \tilde{q})$ to put the saddle point C to the origin of new coordinates $(\tilde{p}, \tilde{q})$ and make coordinate axes $C\tilde{q}$, $C\tilde{p}$ coincide with the main axes of the saddle. Let coordinate axis $C\tilde{p}$ coincide with the vertex defined in section 2. Let $V(\tilde{p}, q, \hat{y}, \hat{x})$ be generating function of transformation $(p, q) \mapsto (\tilde{p}, \tilde{q})$; it contains $\hat{y}, \hat{x}$ as parameters. Initial Hamiltonian function $H(p, q, \hat{y}, \hat{x})$ expressed in the new variables has the form $H = \tilde{H}_0(\tilde{p}, \tilde{q}, \hat{y}, \hat{x})$. Now make canonical transformation $(p, q, \hat{y}, \hat{x}) \mapsto (\tilde{p}, \tilde{q}, \tilde{y}, \tilde{x})$ with generating function $\varepsilon^{-1}\tilde{y}\hat{x} + V(\tilde{p}, q, \tilde{y}, \hat{x})$. The Hamiltonian H in new variables is

$$\tilde{H}(\tilde{p}, \tilde{q}, \tilde{y}, \tilde{x}) = \tilde{H}_0(\tilde{p}, \tilde{q}, \tilde{y}, \tilde{x}) + \varepsilon \tilde{H}_1(\tilde{p}, \tilde{q}, \tilde{y}, \tilde{x}) + \varepsilon^2 \tilde{H}_2.$$

In the system with Hamiltonian function $\tilde{H}_0$ at frozen slow variables $(\tilde{y}, \tilde{x})$ one can introduce action-angle variables $I, \varphi \bmod 2\pi$ with a canonical transformation of variables defined by generating function $W(I, \tilde{q}, \tilde{y}, \tilde{x})$ of the form (3.1). In the new variables $\tilde{H}_0$ has the form $\tilde{H}_0 = H_0(I, \tilde{y}, \tilde{x})$. Now, in the complete system, we make a canonical transformation $(\tilde{p}, \tilde{q}, \tilde{y}, \tilde{x}) \mapsto (\bar{I}, \bar{\varphi}, \bar{y}, \bar{x})$ with generating function $\varepsilon^{-1}\bar{y}\tilde{x} + W(\bar{I}, q, \bar{y}, \tilde{x})$. In the new variables Hamiltonian function H has the form (3.3).

In the following consideration, we have to keep additional terms in H_1 and H_2 (and, accordingly, in $\mathcal{H}_1$ and $\mathcal{H}_2$) generated by the corresponding terms in $\tilde{H}$.

All the formulae before formula (4.31) remain valid if we replace initial slow variables $\hat{y}, \hat{x}$ with $\tilde{y}, \tilde{x}$, function $E = H - h_c(y, x)$ with function $\tilde{E} = \tilde{H}(\tilde{p}, \tilde{q}, \tilde{y}, \tilde{x}) - \tilde{H}_0(0, 0, \tilde{y}, \tilde{x}) = \tilde{H} - h_c(\tilde{y}, \tilde{x})$ and value ξ with $\tilde{\xi} = |\tilde{E}_0/(\varepsilon\Theta_*)|$, where $\tilde{E}_0$ is the value of function $\tilde{E}$ on the trajectory at the moment of the last crossing of the vertex before leaving G_i. Here, we write formula (4.31) as follows:

$$\tilde{\xi} = \frac{1}{2\pi}\left(\varphi_0 + \frac{1}{\varepsilon}\int_0^{\tau_*}(\omega_0 + \varepsilon\omega_1)\mathrm{d}\tau\right) + \frac{1}{\Theta_*}(\mathcal{H}_1)_* + \mathrm{O}(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}) \mod 1. \tag{A.1}$$

In this case

$$\frac{1}{\Theta_*}(\mathcal{H}_1)_* = A_* + \frac{1}{\Theta_*}\tilde{H}_1(0, 0, \tilde{y}, \tilde{x}); \tag{A.2}$$

value A_* was introduced in section 2.

To be definite, consider the case of motion in the domain G_3. In this domain $\tilde{E}_0 > 0$, $\Theta_* > 0$. From (A.1), (A.2) we have

$$\begin{aligned}\frac{1}{\varepsilon\Theta_*}(\tilde{E}_0 - \varepsilon\tilde{H}_1(0, 0, \tilde{y}, \tilde{x})) &= \frac{1}{2\pi}\left(\varphi_0 + \frac{1}{\varepsilon}\int_0^{\tau_*}(\omega_0 + \varepsilon\omega_1)\mathrm{d}\tau\right) + A_* \\ &\quad + \mathrm{O}(\varepsilon^{1/3}|\ln\varepsilon|^{-1/3}) \mod 1.\end{aligned} \tag{A.3}$$

According to the definition of E in section 2, $E = H - h_c(\hat{y}, \hat{x})$. Now, we have

$$\begin{aligned}\tilde{H}(\tilde{p}, \tilde{q}, \tilde{y}, \tilde{x}) &= H(p, q, \hat{y}, \hat{x}), \\ \tilde{H}(0, 0, \tilde{y}, \tilde{x}) &= H(p_c(\hat{y}, \hat{x}) + \mathrm{O}(\varepsilon), q_c(\hat{y}, \hat{x}) + \mathrm{O}(\varepsilon), \hat{y}, \hat{x}) \\ &= H(p_c(\hat{y}, \hat{x}), q_c(\hat{y}, \hat{x}), \hat{y}, \hat{x}) + \mathrm{O}(\varepsilon^2) = h_c(\hat{y}, \hat{x}) + \mathrm{O}(\varepsilon^2).\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{E} - \varepsilon\tilde{H}_1(0, 0, \tilde{y}, \tilde{x}) &= \tilde{H}(\tilde{p}, \tilde{q}, \tilde{y}, \tilde{x}) - \tilde{H}_0(0, 0, \tilde{y}, \tilde{x}) - \varepsilon\tilde{H}_1(0, 0, \tilde{y}, \tilde{x}) \\ &= \tilde{H}(\tilde{p}, \tilde{q}, \tilde{y}, \tilde{x}) - \tilde{H}(0, 0, \tilde{y}, \tilde{x}) + \mathrm{O}(\varepsilon^2) = H(p, q, \hat{y}, \hat{x}) - h_c(\hat{y}, \hat{x}) + \mathrm{O}(\varepsilon^2) \\ &= E + \mathrm{O}(\varepsilon^2).\end{aligned}$$

Thus, (A.3) again implies formula (2.2).

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