Analytical approximation of the emission line

Fe $K_\alpha$ in QSO’s spectra

S.V. Repin,$^1$,*  V.N. Lukash,$^2$,** and V.N. Strokov$^{2,3,***}$

$^1$Space Research Institute of RAS
ul. Profsoyuznaya 84/32, 117997 Moscow, Russia

$^2$AstroSpace Center of the P.N. Lebedev Physical Institute of RAS
ul. Profsoyuznaya 84/32, 117997 Moscow, Russia

$^3$Moscow Institute of Physics and Technology (State University)
Institutskiy per. 9, 141701 Dolgoprudny, Moscow region, Russia

ABSTRACT

In spectra of many Seyfert galaxies there is a wide emission line of Fe $K_\alpha$. The line profile with two maxima supposes that the line emerges in innermost regions of an accretion disk around a black hole, hence, it is necessary to take into account General Relativity (GR) effects. In order to determine GR processes which occur in active galactic nuclei (AGN) an inverse problem of reconstructing the accreting system parameters from the line profile has to be solved quickly. In this paper we present a numerical approximation of the emission line Fe $K_\alpha$ with analytical functions. The approximation is accomplished for a range of the disk radial coordinate $r$ and the angle $\theta$ between line of sight and perpendicular to the disk and allows one to decrease computing time by $10^6$ times in certain astrophysical problems taking into account all GR effects. The approximation results are available in the Internet at http://www.iki.rssi.ru/people/repin/approx.

* Electronic address: repin@mx.iki.rssi.ru
** Electronic address: lukash@asc.rssi.ru
*** Electronic address: strokov@asc.rssi.ru
1. INTRODUCTION

In the past decade quite extensive X-ray observations of Seyfert galaxies have been carried out. In a considerable body of cases the wide emission line of Fe $K_{\alpha}$ ($E_{0} = 6.4$ keV) with a peculiar two-peak profile [1]-[20] is observed in the active galactic nuclei (AGN) of these galaxies. Maxima of the line are of different height, and a long red wing may stretch up to $E \sim 3$ keV. The Doppler line width corresponds to matter velocities of tens of thousand kilometers per second reaching $v \approx 10^{5}$ km/s for the Seyfert galaxy MCG-6-30-15 [2] and $v = 4.8 \cdot 10^{4}$ km/s for MCG-5-23-16 [6]. Today the most reasonable explanation seems the one which indicates that the Fe $K_{\alpha}$ line emerges in inner regions of an accretion disk ($r \sim 1 / 4 r_{g}$) where GR effects dominate. The Fe $K_{\alpha}$ line is also observed in X-ray binaries and $\mu$QSO's. A narrow Fe $K_{\alpha}$ line may also be a part of a wide line, which cannot be distinguished from the background.

In order to determine GR processes which occur in active galactic nuclei (AGN) an inverse problem of reconstructing the accreting system parameters from the line profile has to be solved quickly. To get the line profile one has to solve numerically GR equations of motion for photons with a wide range of initial conditions. These computations are very time-consuming. They have already been carried out before [21]-[35], but an amount of effort is so unusual that it is necessary to simplify essentially the computation procedure. Among other things, one has to decrease the computing time and the amount of computations. In this paper we present an approximation of the Fe $K_{\alpha}$ line profile with analytical functions. The use of these functions offers a decrease of the computing time by $10^{4} - 10^{6}$ times depending on a line parameters. This procedure essentially simplifies the solution of the inverse problem of reconstructing AGN parameters from observational data.

In Sec. 2 we consider the procedure of calculating the Fe $K_{\alpha}$ line profile within the GR framework. In Sec. 3 we propose a method of approximating the Fe $K_{\alpha}$ line profile with analytical functions. Sec. 4 deals with a numerical method (genetic algorithm) of search for optimal approximation parameter values. In Sec. 5 we present the approximation results and estimate its quality. In Sec. 6 we discuss limitations of applying the obtained approximation to the astrophysical problems. The summary is made in Sec. 7.
2. CALCULATING THE LINE PROFILE IN GR FRAMEWORK

Space-time around a rotating black hole is described with the Kerr metrics:

\[
\text{ds}^2 = \left(1 - \frac{r_g r}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{r_g r a^2}{\rho^2} \sin^2 \theta\right) \sin^2 \theta \, d\phi^2 + \frac{2 r_g r a}{\rho^2} \sin^2 \theta \, d\phi dt,
\]

where \((t, r, \theta, \phi)\) are the Boyer-Lindquist coordinates. The standard notations are introduced:

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - r_g r + a^2, \quad r_g = 2GM,
\]

where \(G\) is the gravitational constant, \(a\) and \(M\) are angular momentum and mass of the black hole, respectively. Hereafter we assume almost extreme angular momentum for the black hole \(a \approx 0.9981 M\) and \(G = c = 1\).

Equations of motion of free particles in the Kerr metrics are obtained by separation of variables in the Hamilton-Jacobi equation: \([36-38]\)

\[
p_i p^i = g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} = m^2,
\]

where \(m\) is mass of a particle. For photons \(m = 0\). The Fe \(K\alpha\) line profile registered by a distant observer is obtained by solving equations of photon motion in the Kerr metrics. After we made the set of equations dimensionless we have:

\[
\frac{dt}{d\sigma} = -a \left(a \sin^2 \theta - \xi\right) + \frac{r^2 + a^2}{\Delta} \left(r^2 + a^2 - \xi a\right),
\]

\[
\frac{dr}{d\sigma} = r_1, \quad (2)
\]

\[
\frac{dr_1}{d\sigma} = 2 r^3 + \left(a^2 - \xi^2 - \eta\right) r + (a - \xi)^2 + \eta, \quad (3)
\]

\[
\frac{d\theta}{d\sigma} = \theta_1, \quad (4)
\]

\[
\frac{d\theta_1}{d\sigma} = \cos \theta \left(\frac{\xi^2}{\sin^2 \theta} - a^2 \sin \theta\right), \quad (5)
\]

\[
\frac{d\phi}{d\sigma} = - \left(a - \frac{\xi}{\sin^2 \theta}\right) + \frac{a}{\Delta} \left(r^2 + a^2 - \xi a\right), \quad (6)
\]

where \(\sigma\) is an affine variable, \(\eta\) and \(\xi\) are constants that define a trajectory of a particle. They are expressed through quantities conserving on the trajectory: the particle energy at infinity \(E\), the projection of angular momentum on the \(z\)-axis \((\theta = 0)\) \(L_z\) and the Carter \([36]\)
separation constant\(^1\) \(Q\), viz. \(\eta = Q/M^2E^2\) and \(\xi = L_z/M.E\). In the set of equations (1)-(6) the angular momentum \(a\), the coordinates \(t\) and \(r\) are measured in the units of the black hole mass. Respectively, \(\Delta\) measured in the units \(M^2\) is related to the dimensionless \(r\) and \(a\) as follows: \(\Delta = r^2 - 2r + a^2\). The extra variables \(r_1\) and \(\theta_1\) are introduced for the set not to have singularities, and their physical meaning is not important to us. Two first integrals of the system

\[
\epsilon_1 \equiv r_1^2 - r^4 - (a^2 - \xi^2 - \eta) r^2 - 2 [(a - \xi)^2 + \eta] r + a^2 \eta = 0, \tag{7}
\]

\[
\epsilon_2 \equiv \theta_1^2 - \eta - \cos^2 \theta \left( a^2 - \frac{\xi^2}{\sin^2 \theta} \right) = 0, \tag{8}
\]

are used to control the calculation accuracy and avoid accumulation of integralional errors. Namely, quantities \(\epsilon_1\) and \(\epsilon_2\) have to be smaller than \(10^{-8}\) in the end of the trajectory. The method of solving equations (1)-(6), results of simulations, as well as the derivation of the equations are available in papers [28, 39, 40].

To solve the set (1)-(6) numerically one has to set initial conditions. We assume that particles in the disk move along circular orbits. The disk is in an equatorial plane, optically thick and radiates monochromatic photons with the energy \(E_0\) isotropically in the laboratory reference frame comoving with the disk. We assume the disk to be opaque, i.e. the disk absorbs a photon crossing the disk plane. In other words, a photon emitted from one side of the disk cannot be registered from the other side. A profile of the line radiated by a thin ring with \(r = r_0 = \text{const}\) is obtained by setting initial conditions at \(r = r_0\) and collecting at infinity the photons, which come to a distant observer in the direction \(\theta\). A profile of the line radiated by the entire disk can be obtained by integrating the intensity with respect to the radial coordinate.

Thus, given the disk and the direction \(\theta\), the line profile normalized at the unity in the maximum is the dependance of the intensity on the photon energy at infinity. The energy is measured in the units of the laboratory energy \(E_0\):

\[
K_\alpha = K_\alpha (x) \tag{9}
\]

where

\[
x = \frac{E}{E_0}. \tag{10}
\]

\(^1\) \(Q = p_\theta^2 + \cos^2 \theta \left( \frac{L_z^2}{\sin^2 \theta} - a^2M^2E^2 \right)\), where \(p_\theta\) is the \(\theta\)-component of 4-momentum.
Figure 1. Typical Fe $K_\alpha$ line profile. Radial coordinate value $r = 4r_g$, angle between line of sight and perpendicular to the disk $\theta = 30^\circ$. Photon energy in reference frame comoving with the disk is taken for unity. Small-scale fluctuations are due to statistical reasons.

Figure 2. Plot of function (12) with $a_1 = 0.011181$, $\alpha_1 = 1.73588$, $a_2 = 0.000194$, $b_2 = 85.16571$, $\alpha_2 = 0.731736$, $X_1 = 0.6711022$, $X_2 = 1.005738$. Parameter values correspond to approximation of the curve shown in Fig. 1. In this Fig. normalization is not taken into account.

A typical Fe $K_\alpha$ line profile registered by a distant observer with $\theta = 30^\circ$ from a ring of the radius $r_0 = 4r_g$ is shown in Fig. 1.

3. LINE APPROXIMATION

If we take a look at the typical Fe $K_\alpha$ line profile (see Fig. 1) we notice that each maximum has both sharp and gentle slope. Therefore, it is necessary to find an analytical function with this property.
As it is well-known from mathematical analysis, the function \( f(x) = \exp(-a/x) \) tends to zero as \( x \to 0 \), moreover, it tends faster than any power-law function does. Besides, \( f(x) \to 1 \) as \( x \to \infty \). The power-law function \( g(x) = x^{-\alpha}, \alpha > 0 \), on the contrary, tends to infinity as \( x \to 0 \), and to zero as \( x \to \infty \). Then their product

\[
y(x) = f(x) \cdot g(x) = \exp \left( -\frac{a}{x} \right) \cdot \frac{1}{x^\alpha}
\]

has all the required properties. On one hand, if \( a \) is large enough the latter function has a sharp slope (since the function behavior is \( \exp(-a/x) \) if \( a \) is large), on the other hand the function has a gentle power-law slope (since \( \exp(-a/x) \sim 1 \) if \( x \) is large).

Writing the similar terms for each of the two maxima of the line we obtain an approximation formula:

\[
y(x) = e^{-\frac{a_1}{x - X_1}} \cdot \frac{1}{(x - X_1)^{\alpha_1}} + e^{-\frac{a_2}{X_2 - x}} \cdot \frac{b_2}{(X_2 - x)^{\alpha_2}}, \quad X_1 < x < X_2,
\]

which comprises seven parameters. The parameters \( a_1 \) and \( a_2 \) determine sharpness of the line maxima, \( X_1 \) and \( X_2 \) determine approximate positions of both maxima, \( \alpha_1 \) and \( \alpha_2 \) determine the function behavior between the maxima, and \( b_2 \) sets the relative heights of the maxima. The coefficient \( b_1 \) may be set equal to unity. An example of a non-normalized plot of the function is shown in Fig. 2.

4. NUMERICAL METHOD. GENETIC ALGORITHM.

The approximation quality can be estimated by calculating a sum of squared deviations of the function (12) from a line profile, which is obtained by solving the set (1)–(6) and shown in Fig. 1. In other words, to estimate the approximation quality it is required to find a minimum of the function:

\[
F = \sum_{i=n_1}^{n_2} \left[ \frac{y(x_i)}{y_{\max}} - K_{\alpha}(x_i) \right]^2,
\]

in the 7-dimensional parameter space \((a_1, X_1, \alpha_1, a_2, X_2, b_2, \alpha_2)\). Here \( x_i \) are selected points of the normalized energy (10), \( y(x_i) \) is the approximation from (12), \( y_{\max} \) is the maximal value from numbers \( y(x_i) \), \( K_{\alpha}(x_i) \) is the solution (9) obtained from the set (1)–(6), and \( n_1 \) and \( n_2 \) are the minimal and maximal index values when \( K_{\alpha}(x_i) \neq 0 \) (one can see in Fig. 1 that on the right and left sides the plot has zero segments; accurate meaning of the
variables $n_1$ and $n_2$ is to be cleared below). In such cases a domain of the 7D space is usually covered with a grid and the function $F$ values are calculated in nodes of the grid. However, it is practically impossible to accomplish the task with exhaustive search. For minimally acceptable approximation accuracy the grid should be fine enough: from 0 to 2 with step 0.01 in $(a_1, a_2)$, from 0 to 2.5 with step 0.01 in $(\alpha_1, \alpha_2)$, from 0 to 200 with step 1 in $b_2$. The values $(X_1, X_2)$ are known approximately (they are positions of the maxima), however, even close to these values a grid with at least 100 nodes is required. Thus, for the acceptable approximation the whole number of the grid nodes (possible parameter combinations) is over $10^{15}$, which is beyond computing capability.

For multidimensional problems like this one may apply the genetic algorithm, efficiency of which increases with an increase of number of dimensions. For the algorithm to work efficiently just continuity and relative smoothness is required. Using the algorithm does not guarantee that we find the exact minimum, however, this algorithm finds this minimum with quite high probability.

The idea of the algorithm is taken from biology: the fittest survive, the weakest die. The domain of the multidimensional space is first covered with a sufficiently fine grid, and nodes of the grid are encoded with consequent binary natural numbers in each coordinate. This binary number with a certain number of zeros and units (bits) is called a gene. (e.g. 00101101). Each coordinate gets its own number of bits. For instance, if the grid has 256 nodes with respect to some dimension, the genes corresponding to this dimension contain 8 bits (one of the genes, viz. the 46th, is mentioned above). The number of genes for each node is the same as the number of the space dimensions. A chromosome of a certain node is a system of the genes which are written consequently (in the parameters' order) without spaces (e.g. if 010 and 1001 are genes then 0101001 is a chromosome). The number of bits in a chromosome is equal to the sum of bits of the constituent genes. Value of a chromosome is defined as the value of the function $-F$ in a node corresponding to this chromosome.

The essence of the algorithm is the technique with which the chromosome with maximal value, i.e. the minimum of the function $F$, is searched. At the initial instant some number of chromosomes is randomly selected in the multidimensional space and put in descending order of their values. Then the operation of crossover is performed. To do this pairs of chromosomes are randomly selected; the higher a chromosome value the higher the probability to crossover. Crossover is an operation of cutting each of the two chromosomes at a random point and
exchanging the cut parts. For the newly born chromosomes their values are calculated. Then
the entire population, parents and their off-spring, are put in descending order again, and
the weakest chromosomes are omitted so that the number of chromosomes remains the same.
After several generations the fittest chromosome corresponds with high probability to the
minimum of the function $F$.

Note that just like in biology chromosomes can mutate. Mutation is a random inversion
of a bit in a chromosome. To fulfill the procedure we make the chromosome mutate after
crossover, but before calculating its value. Namely, consequently scanning all its bits we
invert each of them with very small probability (0.001 or smaller). If a generation contains
200 chromosomes and each is 20 bits long then after mutation 4 bits (of $20 \cdot 200 = 4000$) will
be inverted. This procedure allows one to refresh the generation and speed up the minimum
search. See e.g. [44, 45] for detailed description of the genetic algorithm.

The function (13) is calculated at $n$ fixed points $x_i$ uniformly distributed in logarithmic
scale. We should first choose an interval between the points

$$W = \frac{1}{n} \left( \log E_H - \log E_L \right) = \frac{1}{n} \cdot \log \frac{E_H}{E_L},$$

where $E_L$ is the minimal value of the energy, $E_H$ is the maximal value of the energy, $n$ is
number of the intervals, and fix points of the interval separation

$$\log \xi_k = \log E_L + kW, \quad k = 0, 1, \ldots, n. \tag{15}$$

The points $x_i$, in which we calculate values of the function $F(x)$, are in the middle between
the separation points:

$$x_{k+1} = \frac{\xi_k + \xi_{k+1}}{2} = \frac{1 + 10^W}{2} \cdot 10^{\log E_L + kW}, \quad k = 0, 1, \ldots, n - 1. \tag{16}$$

The values of the variables $E_L$ and $E_H$ should be chosen in such a way that the spectral line
were within the segment $[E_L, E_H]$. Even better if the segment covers the zero segments as
well. To calculate values of the function $F$ (13) not all values $x_i$ are required, just the ones
where $K_a(x_i) > 0$. In other words, if $n_1$ and $n_2$ stand for the minimal and maximal values
of $k$ where $K_a(x_i) > 0$ we use them in (13) as summation limits.

Thus, the function values are calculated only at the aforementioned points $x_i$, and we let
$F(x) = 0$ as $x < x_{n_1}$ and $x > x_{n_2}$. To perform the calculations the function $y(x)$ should be
normalized by dividing by the maximal value at the points $x_i$. The $F(x)$ function behavior
between the points $x_i$ is not considered. Among other things, one cannot guarantee that the functions $F(x)$ and $K_{\alpha}(x)$ have the maximum at the same point. Note that the interval width $W$ in (14) also determines the precision of the function value localization of $F(x)$ and $K_{\alpha}(x)$ on the $x$-axis. In other words, the values of these functions are considered to be constant between the points $\xi_k$ and $\xi_{k+1}$.

The minimum of the function (13) was searched at several stages. At first, the minimum of $F(x)$ was found 10 times with the constant parameter intervals on a quite rough grid (there were 1000 chromosomes and 80 generations). The best result of the ten was used to center the domain in the 7-dimensional space. After that the minimum search was performed again, but on a finer grid. Then the process was executed two more times. The result of the last iteration was accepted as the minimum of $F(x)$. Note that such a cumbersome procedure is required, because the function $F(x)$ has a number of accessory minima which are hard to distinguish from the main minimum (this is the reason why gradient methods are not applicable here). Using the genetic algorithm does not guarantee that we find the "very" main maximum. However, from our experience we can say that even if in some cases the genetics was able to find only an accessory maximum it yielded an adequate approximation.

5. SIMULATION RESULTS

In Fig. 3 the most typical results of the approximation for two sets of initial parameters $r = 4.5 r_g$, $\theta = 30^\circ$ and $r = 3.4 r_g$, $\theta = 60^\circ$ are shown. The initial theoretical curve obtained by solving numerically the set (1)–(6) is plotted with a thin line, and the result of its approximation is plotted with a thick line. Theoretical curves in this and other cases are wavy, but these fluctuations are statistical [40] (shot noise) and not physical. From Fig. 3 one can see that the theoretical and approximation lines are quite close to each other and cannot be distinguished with a naked eye.

The best approximation results are impossible to be shown on the plot at reasonable scale, because the discrepancy between the curves is about a thin line thick. One can imagine that one of these plots (with two curves) is shown in Fig. 1 or 2, but because of the tiny discrepancy between the curves we cannot see them separately. For $r = 4.2 r_g$ and $\theta = 60^\circ$ the curves deviate from each other at any point less than 0.5% from the maximal value.

The results of approximating the line Fe $K_{\alpha}$ for the fixed value of the radial coordinate
Figure 3. Typical result of approximating the line Fe $K_{\alpha}$. The initial theoretical curve is plotted with a thin line, and the result of its approximation is plotted with a bold line. The values of the radial coordinate $r$ and an angle $\theta$ between line of sight and perpendicular to the disk are written on each plot.

Figure 4. Approximation results for the fixed value of the radial coordinate $r = 3.4 r_g$ and different values of an angle between line of sight and perpendicular to the disk from $\theta = 10^\circ$ to $\theta = 80^\circ$. $x$-axis scale is different for different plots. and different values of the disk inclination angle are shown in Fig. 4. One of the plots
Figure 5. Approximation results for the fixed value $\theta = 40^\circ$ of an angle between line of sight and perpendicular to the disk and different values of the radial coordinate from $r = 0.8r_g$ to $r = 10r_g$. x-axis scale is different for different plots. with $\theta = 60^\circ$ shown in Fig. 3 can also be added herein. As one can see from the plots the approximation is adequate for all disk inclination angles up to $\theta = 80^\circ$. In the latter case ($\theta = 80^\circ$) the proposed approximation model may become inadequate, because the lensing effects play a big role, and one or two extra maxima [46] may appear at some values of $r$. However, it is possible to apply the approximations for practical problems which do not require high accuracy. If one wants to obtain a more accurate approximation for curves with $\theta = 80^\circ$ one can add to the expression (12) a quadratic function with a negative leading coefficient. This extra term will describe an extra detail in the spectrum (a detail similar to that in Fig. 4 for $\theta = 80^\circ$ close to $E = 1.15$). For $\theta \geq 85^\circ$ the proposed model is inadequate.

In Fig. 5 the approximation results for the fixed value of the disk inclination angle $\theta = 40^\circ$ and different values of the radial coordinate $r$ are shown. As one can see from the plots the approximation is adequate for a wide range of the radial coordinate values. Note that the approximation remains adequate at the very boundaries of the interval, i.e. for $r = 0.8r_g$ and $r = 10r_g$, and some statistical fluctuations at $r = 0.8r_g$ do not make it worse. Extrapolating the results we can guess that even for $r > 10r_g$ the approximation (12) is quite reliable.
Figure 6. Approximation quality. Root-mean-square \( z_1(r, \theta) \) in significant part of the curve measured in percentage. \( x \) and \( y \) axes are variables \( r \) and \( \theta \), respectively.

In Tab. 1 values of the parameters \( a_1, \alpha_1, a_2, b_2, \alpha_2, X_1, X_2 \) are given for some values of the radial coordinate \( r \) and the disk inclination angle \( \theta \). Some technical information, for which the parameter values were obtained, are given in Tab. 2. The detailed information on all the parameter values for \( 0.7 r_g < r < 10 r_g \) and \( 10^\circ < \theta < 80^\circ \) is available in the Internet at [http://www.iki.rssi.ru/people/repin/approx](http://www.iki.rssi.ru/people/repin/approx).

To estimate the approximation quality beside the function (13) we also used

\[
z_1(r, \theta) = \sqrt{\frac{\sum_{i=n_1}^{n_2} \left[ \frac{y(x_i)}{y_{\text{max}}} - K_\alpha(x_i) \right]^2}{n_2 - n_1 + 1}},
\]

where \( n_1 \) and \( n_2 \) are the numbers of the first and last intervals in a significant part of the curve, respectively. This value is a root-mean-squared per interval in a significant part of the curve. The term "significant" means that we consider only the intervals with at least one photon. For instance, for \( r = 3.4 r_g \) and \( \theta = 20^\circ \) the calculations were carried out for
\( E_L = 0.2, \ E_H = 1.4, \ n = 150. \) The line itself occupies the interval \( 0.66 < x_i < 0.92. \) The statement made above means that while calculating the root-mean-square the intervals with \( x_i < 0.66 \) and \( x_i > 0.92 \) were omitted.

It is convenient to measure the function \( z_1 (r, \theta) \) in percentage. The plot of this function is shown in Fig. 6. The values 0.2 - 0.5% are typical. The best approximation results are for \( 1.5 r_g < r < 6 r_g. \) For higher and lower values of \( r \) the approximation is some worse, but even though the deviations higher than 0.7% are quite rare. For stand-alone points with \( r < r_g \) and \( r > 6.5 r_g \) a more accurate approximation will be found some day.

To estimate the approximation quality one can use an extra criterion which describes the maximal discrepancy between theoretical and approximation curves

\[
z_2 (r, \theta) = \max \left| \frac{y(x_i)}{y_{\text{max}}} - K_n (x_i) \right|, \quad i = 1, \ldots, n.
\]

and measured in percentage of the maximal value of a curve. The plot of the value is shown in Fig. 7. As one can see from the plot, the maximal discrepancy is mainly 6-8%, and discrepancies higher than 10% are quite rare. Besides, this criterion has no explicit dependance on variables \( r \) and \( \theta. \) Intervals with maximal discrepancies are most often located close to the blue (higher) maximum from its left side where the curve drops sharply. Usually it is one or two points. Thus, this criterion characterizes "the worst" point of the curve.

6. DISCUSSION

As it has been noticed above, the function \( F(x) \) values can be taken only at the points \( x_i \) defined by the relations (14) and (16), because between these points the function \( F(x) \) values are not defined. Since it also applies to the points \( x_i \) close to the maxima the used approach does not guarantee that the function \( F(x) \) reaches its maximum at the same point as \( K_n (x). \) When using the approximation (12) (e.g. to model a profile of the line radiated by the entire disk \([41-43]\)) one should first calculate intensities at the points \( x_i \) for several thin rings and then sum up the results with a decrease of number of intervals in the resulting curve (hence, make it rougher). This procedure allows one to obtain a quite smooth curve which can be used to interpret observational data.

In Fig. 8 examples of the approximations with the highest deviations from the theoretical curves are shown. Although these cases are rare, we find it important to explain reasons
which caused them. One of the reasons is improper photon energy distribution in higher (blue) and lower (red) maxima. As a result, a maximum consists of two values with almost the same height which is not suitable for the approximation \( (12) \). This situation occurs for the second (red maximum) and third (blue maximum) plots in Fig. 8. This problem can be solved by changing the location of the points \( x_i \) close to the maxima which, in turn, can be achieved by changing values of \( E_L, E_H \) and \( n \). In this way one can obtain approximations of higher quality for the curves shown in Fig. 8, but this is quite time-consuming. The other reason appears at the small values \( r < 1.2 r_g \). Here in a quite wide range of the disk inclination angle \( \theta \), as a matter of fact, a curve has just one distinct maximum and the approximation \( (12) \) does not work.
Figure 8. Examples of approximations with the highest deviation from theoretical curves. Values of radial coordinate $r$ and angle $\theta$ between line of sight and perpendicular to the disk are written on each plot.

7. CONCLUSIONS

We obtained an analytical approximation of the iron emission line profile in AGN, which allows one to determine physical parameters of the matter of the accreting disk around a black hole by the line profile.

According to the criterion (17) in the intervals $1.3 r_g < r < 2.5 r_g$ with respect to the radial coordinate and $10^\circ < \theta \leq 80^\circ$ with respect to the angle $\theta$ as well as $3.5 r_g < r < 6 r_g$, $20^\circ < \theta \leq 80^\circ$ the approximation (12) yields the accuracy of 0.8% excluding several points
with higher deviation, but still lower than 1.4%. In the interval \(0.7 r_g < r \leq 1.3 r_g\) for all values of \(\theta\) one has the accuracy of 1.2% excluding two points with 1.6%.

According to the criterion of the maximal discrepancy (18) the approximation (12) yields the accuracy from 2% to 12% in the entire investigated interval \(0.7 r_g < r < 10 r_g\). No explicit dependance of the approximation accuracy on \(r\) and \(\theta\) was noticed. As an exception there are less than ten points (mainly at \(r < 4 r_g\)) in which the discrepancy amounts to 18%.

The application of the approximation (12) in practical astrophysical problems results in a decrease of computing time by \(10^4 - 10^6\) times.

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**Table 1.** Approximation results for some values of radial coordinate $r$ and angle $\theta$ between line of sight and perpendicular to the disk.
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**Table 2.** Technical information for the same values of radial coordinate $r$ and angle $\theta$ between line of sight and perpendicular to the disk as in Tab.1.
REFERENCES