RIGID BODY ROTATION EVOLUTION DUE TO A DISTURBING TORQUE WHICH IS KNOWN IN A BODY FRAME

M.L.Pivovarov, Moscow, Russia, **L.O.Ferreira** and **R.V.F.Lopes,** S.J.Campos, Brazil

Summary. This paper deals with the problem of rigid body rotation evolution under the action of a torque which depends on the projections of rigid body angular velocity onto the body frame only. This is the so called *rigid body with self-excitation*. If the magnitude of the torque is arbitrary, the solution to the problem is known in the simplest cases only. The problem with small self-excitation is considered in the paper. This problem is of great importance from the standpoint of a number of applications in spacecraft and rigid body dynamics. Perhaps the most important example is spin-stabilized spacecraft rotation evolution due to an internal disturbing torque. In the case of a small torque, it is possible to use perturbation methods to obtain complete analytical and geometrical descriptions of the rotation. In this paper, the common approach to investigating the problem of rigid body rotation under the action of a small torque known in the body frame is described. Using this approach, two problems are solved: the problem of maintaining the angular velocity of a gyroscope using a control torque quadratic in the angular velocity (the problem of Grammel for the case of a small torque), and the problem of asymmetric spacecraft rotation evolution with an electric propulsion thruster.

1. Introduction

Grammel [1] and Magnus [2] call "gyroscope with self-excitation" a rotation of a rigid body under the action of a torque which is known in a body frame. In other words, the torque depends on the components of angular velocity and time only. This problem is original, since the equations of motion are ucoupled in some sense. Namely, the motion in a body frame can be investigated independently of the body motion in an inertial frame. Grammel presented the problem of maintaining the angular velocity of a gyroscope by using a control moment proportional to the difference between the squares of the actual angular velocity and the required angular velocity.

If the magnitude of a torque is arbitrary, closed-form solution of the problem of rigid body rotation with self-excitation is known in some simple cases only. For example, when the body is axially symmetric and the torque is directed along a principal axis of inertia. The problem of rigid body rotation with small self-excitation is of great importance from two points of view. First, the assumption that the torque is small compared to the kinetic energy of rotation is reasonable in a number of applications in spacecraft attitude dynamics and rigid body dynamics. Second, it allows one to carry out analytical investigation using asymptotic methods of nonlinear mechanics.

Maybe the best example of a practical problem which can be treated as rigid body rotation with small self-excitation is rotation evolution of a spacecraft equipped with a small thruster for orbit control. The offset between the direction of thrust and the center of mass position gives rise to a disturbing torque which is constant in a body frame. Perhaps the first results about this problem (axially symmetric case) were those obtained by Beletsky [3]. Three-axis rigid body rotation evolution with small self-excitation has been studied in the papers: Neishtadt and Pivovarov [4] (constant-torque case), Neishtadt [5] (the torque is a sum of constant terms with terms which are linear in the angular velocity), and Pivovarov [6] (the torque includes terms quadratic in the angular velocity).

Thanks to the results of Chernous'ko [7-9], it is known that the motion of a rigid body containing a cavity filled with a strongly viscous fluid (fuel in tanks) can be asymptotically described as the action of a torque which is homogeneous cubic polynomial in the components of the rigid body angular velocity. The coefficients of the polynomial depend on the shape of the cavity. Also Chernous'ko carried out asymptotic analyses of the dynamics of a rigid body with internal masses which are connected with the body by elastic and viscous constraints. It has been shown that if stiffness and viscosity of constraints are strong, the rotation of the rigid body can be asymptotically described as the action of a torque which is homogeneous fourth- and fifth- degree polynomial in the components of the angular velocity. The coefficients of polynomials depend on the geometry of the constraints. The most important application of this problem is in nutation damping of spin-stabilized spacecraft using different kinds of dampers. That is why the problem of self-excited rigid body rotation is important from the standpoint of nonrigid body dynamics, too.

The problems of rigid body rotation evolution under the action of a disturbing torque which is known in the body frame can be investigated using a common approach.

We will describe this approach and use it for solving the following problems:

1. The problem of Grammel in the case of small self-excitaion;
2. The spacecraft rotation evolution with an electric propulsion thruster.

2. The problem of Grammel

The equations of the problem are:

$$
\begin{aligned}
&A dp/dt + (C - B)qr = 0, \\
&B dq/dt + (A - C)rp = 0, \\
&C dr/dt + (B - A)pq = \varepsilon(r_0^2 - r^2).
\end{aligned}
\tag{1}
$$

Here, A, B, C are the principal central inertia moments of the body; p, q, r are the projections of the angular velocity on the main axes; r_0 is the required angular velocity of the gyroscope; and ε is a small parameter. We will also denote A, B, C the corresponding axes of the inertia ellipsoid. We assume that unperturbed rotation ($\varepsilon = 0$) is the rotation "around" C axis of the inertia ellipsoid; in other words, the r projection preserves sign and is positive.

The goal of the control (the torque on the right-hand side of the system) is to maintain the required gyroscope angular velocity r_0. The analysis of the problem without restriction on the magnitude of the torque is presented by Leimanis [10]. It turns out that it is possible to have a picture of the motion in some cases only. We will consider the problem of Grammel under the assumption that control torque is small compared to the rigid body rotational kinetic energy.

The unperturbed — Euler-Poinsot — problem has the following integrals for energy T and angular momentum G:

$$
\begin{aligned}
2T &= Ap^2 + Bq^2 + Cr^2, \\
G^2 &= A^2p^2 + B^2q^2 + C^2r^2.
\end{aligned}
\tag{2}
$$

Assume $C > B > A$. Then, the general solution for the unperturbed problem has the form:

$$
\begin{aligned}
p &= \sqrt{\frac{2CT-G^2}{A(C-A)}}\,\mathrm{cn}(\tau,k), \qquad q = \sqrt{\frac{2CT-G^2}{B(C-B)}}\,\mathrm{sn}(\tau,k),\\
r &= \sqrt{\frac{G^2-2AT}{C(C-A)}}\,\mathrm{dn}(\tau,k), \qquad \tau = t\sqrt{\frac{(C-B)(G^2-2AT)}{ABC}},\\
k^2 &= \Big[(B-A)(2CT-G^2)\Big] \Big/ \Big[(C-B)(G^2-2AT)\Big],
\end{aligned}
\tag{3}
$$

where k is the modulus of elliptic Jacobi functions $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$. (It is assumed that, at the initial time, $q = 0, q' > 0$).

We shall find the solution of the input problem (1) in the form (2), where T, G are new variables. We obtain the following differential equations for T and G:

$$
\begin{aligned}
dT/dt &= \varepsilon(r_0^2 - r^2)r,\\
dG^2/dt &= 2\varepsilon C(r_0^2 - r^2)r.
\end{aligned}
\tag{4}
$$

We shall use the method of averaging, Sanders and Verhulst [11]. According to this method, we shall average the right-hand sides of eqs. (4) along the unperturbed motion with respect to time. Denote

$$
< f > = \frac{1}{2K}\int_0^{2K} f(x)\,dx,
$$

where $2K$ is the period of the unperturbed motion and K is the complete elliptic integral of the first kind with modulus k. Averaging the eqs. (4), we obtain

$$
\begin{aligned}
dT/dt &= \varepsilon(r_0^2 < r > - < r^3 >),\\
dG^2/dt &= 2\varepsilon C(r_0^2 < r > - < r^3 >).
\end{aligned}
\tag{5}
$$

Using, for example, Byrd and Friedman [12], we can find that

$$
< r > = \frac{\pi}{2K}\sqrt{\frac{G^2-2AT}{C(C-A)}}, \qquad < r^3 > = \frac{\pi(2-k^2)}{4K}\left\{\frac{G^2-2AT}{C(C-A)}\right\}^{3/2}.
\tag{6}
$$

It is clear that eqs. (5) have the integral

$$
H = 2CT - G^2
\tag{7}
$$

and hence are integrable by quadrature.

Remark. Let us consider Euler dynamical eqs. (1) with right-hand sides $\varepsilon M_1(p,q,r)$, $\varepsilon M_2(p,q,r)$, and $\varepsilon M_3(p,q,r)$. Let M_3 be an arbitrary function, but $< pM_1 > = < qM_2 > = 0$. (For example, $M_1 = \text{const}$, $M_2 = \text{const}$.) Then, the averaged system has the integral (7) and is integrable by quadrature. Rewriting (7) in the form

$$
C(C-A)p^2 + B(C-B)q^2 = H,
$$

we see that in the p, q, r space the trajectory remains on the surface of an elliptic cylinder whose axis coincides with the C axis of the inertia ellipsoid.

Let us determine singular points of system (4). The equation of the singular curve has the form $< r^3 > / < r >= r_0^2$. Using (3) and (6), one can see that the singular curve is a straight line:

$$G^2 = 2DT + d, \qquad (8)$$
$$D = \frac{AC + BC - 2AB}{2C - A - B}, \qquad d = 2r_0^2 \frac{C(C-A)(C-B)}{2C - A - B}.$$

Note that $A < D < B$, $d > 0$, and D does not depend on the disturbing torque.

Figure 1 shows a phase portrait of the averaged system (5). The letters A, B and C denote rays corresponding to permanent rotations about axes A, B, C, respectively. The equations of permanent rotations are $G^2 = 2AT$, $G^2 = 2BT$ and $G^2 = 2CT$, respectively. H and H^* are rays (7) for two different values of the constant of the integral. The line marked by D represents singular line (8) for some value of the parameter r_0. The region of unperturbed motion under consideration corresponds to the part of the plane lying between rays B and C. In view of the inequalities $2AT \le G^2 \le 2CT$, no G^2 and T correspond to the shaded region of the plane. Using (7) it is easy to obtain, from (8), that, if $2r_0^2 C(C-A)(C-B)/(H(B-A)) > 1$, then there is a unique equilibrium position of system (4) on ray H. Otherwise (ray H^*), there are no singular points on the ray. The direction of motion of the phase points is indicated by arrows in Fig. 1. Along the rays H and C, the phase point tends to the equilibrium position while, along H^*, it crosses the separatrix of the unperturbed problem. So, the presented approach allows one to carry out the complete analytical and geometrical analysis of the problem in the case of small disturbing torque.

3. Rotating spacecraft jet damping

We shall now consider spin-stabilized spacecraft/rocket attitude motion with a thruster. It is assumed that the direction of thrust coincides with the spacecraft principal inertia axis which, in the ideal case, coincides with the spin direction. The thruster provides the control of spacecraft orbital motion.

It is known, Magnus [2], that the thrust causes a disturbing torque (in spite of the fact that direction of thrust goes through the center of mass of the system!). The reason is that this system is not rigid; the moving particles inside the rotating body cause the torque which is orthogonal to the thrust direction. It turns out that this torque is “useful”, because it leads to a decrease in rigid body nutation (rotating rocket jet damping). Magnus [2] presented analytical solution to the problem for an axially symmetric rigid body without restriction on the magnitude of the thrust.

We shall consider asymmetric spacecraft with an electric propulsion thruster rotation evolution. First, such kinds of thrusters are are being very actively developed and will be widely used soon (Solar electric propulsion thrusters; USA, Russia). Second, spacecraft rotation evolution with this kind of thruster can be investigated using the approach presented above. Denote F the thrust, $\dot{m}$ mass flow, V exhaust velocity. Then, $F = \dot{m}V$. The main peculiarity of this type of thruster is that $\dot{m}$ is extremely small (about $5 \cdot 10^{-3} - 1 \cdot 10^{-2}$ g/s), but thanks to extremely high V (about 15-30 km/s) the thrust is significant.

The equations of motion, Magnus [2], are:

$$\begin{aligned} &A dp/dt + (C - B)qr = -\varepsilon S p, \\ &B dq/dt + (A - C)rp = -\varepsilon S q, \qquad (9) \\ &C dr/dt + (B - A)pq = 0, \end{aligned}$$

where S is a positive constructive constant characteristic of the thruster. A small parameter ε indicates that the disturbing torque due to electric propulsion is small as compared to the spacecraft rotational kinetic energy. It is supposed that the unperturbed spacecraft attitude motion is rotation "around" C axis in the same sense as in the problem of Grammel. The thrust is directed along the C axis and is negative. We assume that $C > B > A$.

We shall use the same approach as the one used in the problem of Grammel. The averaged equations have the form

$$\begin{aligned} dT/dt &= -\varepsilon S(< p^2 > + < q^2 >), \\ dG^2/dt &= -2\varepsilon S(A < p^2 > + B < q^2 >). \end{aligned} \tag{10}$$

Using Byrd and Friedman [12], one can obtain

$$\begin{aligned} < p^2 > &= \frac{(E - Kk'^2)(G^2 - 2AT)(C - B)}{KA(C - A)(B - A)}, \\ < q^2 > &= \frac{(K - E)(G^2 - 2AT)}{KB(B - A)}, \end{aligned} \tag{11}$$

where K,E are the complete elliptic integrals with modulus k, given in (3), and $k'^2 = 1 - k^2$.
Remark. The inertia moments A, B and C in eqs. (9) depend on time, because the mass of the system decreases with time. We omitted the terms with derivatives of inertia moments with respect to time. The mass flow $\dot{m} \ll 1$, which is why the terms with derivatives of inertia moments can be neglected even if we investigate the rotation evolution over a time interval which is much longer than the period of rigid body rotation.

In order to analyze the behaviour of the solutions of system (10), we introduce the function H:

$$H = G^2 - 2CT \leq 0$$

The H function is negative in the region of motion (the part of the phase plane between rays B and C) and is equal to zero on the C ray only. It is easy to see that the total time derivative of the H function with respect to the averaged system (10) (and also the input system (9)) is positive:

$$\frac{dH}{dt} = 2\varepsilon S \left[(C - A) < p^2 > + (C - B) < q^2 >\right] > 0$$

It means that system (10) has no singular points inside the region of the motion and each trajectory tends to the C ray, which corresponds to a permanent rotation around C axis. In other words, the jet of the thruster damps spacecraft nutation. The process of damping is described by eqs. (10), (11).

From eqs. (10), (11) we obtain

$$\begin{aligned} \frac{dT}{dt} &= -\varepsilon S \frac{G^2 - 2AT}{B - A} \left(\frac{E - Kk'^2}{KA} \frac{C - B}{C - A} + \frac{K - E}{KB} \right), \\ \frac{dG^2}{dt} &= -2\varepsilon S \frac{G^2 - 2AT}{B - A} \left(\frac{E - Kk'^2}{K} \frac{C - B}{C - A} + \frac{K - E}{K} \right). \end{aligned} \tag{12}$$

As the separatrix is approached, $K \to \infty$, $E \to 0$, and from eqs. (12) one can find that $dG^2/dT \to 2B$. It means that all trajectories have B ray as the tangent (on the separatrix, $G^2 = 2BT$).

The modulus of elliptic functions, k, is small outside a small vicinity of the separatrix. Using an expansion of the complete elliptic integrals in a series within k [12], we have

$$\left(E - Kk'^2\right)/K = k^2/2 + O(k^4), \qquad (K - E)/K = k^2/2 + O(k^4). \tag{13}$$

Neglecting the terms of order k^4 in eqs. (13), we finally have the approximate integral of eqs. (12):

$$G^2 - 2\alpha T = \text{const}, \qquad \alpha = \frac{AB(2C - A - B)}{C(A + B) - A^2 - B^2}. \tag{14}$$

It is easy to show that $0 < \alpha < A$. Figure 2 shows the phase portrait for the problem.

Using eqs. (3) and (12), we can obtain

$$\frac{dk^2}{dt} = -2\varepsilon S\left[\frac{E - Kk'^2}{KA} + \frac{(K - E)k'^2}{KB}\right]. \tag{15}$$

Taking eq. (14) into account, it is easy to get an approximate analytical solution to eq. (15):

$$k^2 = k_0^2 \exp\left[(-\varepsilon St/(A + B)\right], \tag{16}$$

where k_0^2 is the initial value of the modulus. So, eqs. (3), (14), (16) give an approximate closed-form solution to problem (10).

It is clear that the input problem (9) has a singular line: $p = q = 0$, $r = r_0$. Using Liapunov's direct method, Meirovitch [13], we now show that each equilibrium point is stable. We introduce a Liapunov function V in the form

$$V = 2CT - G^2 + (G^2 - C^2 r_0^2)^2, \tag{17}$$

where T and G^2 are defined by eq. (2). Evidently, the V function is equal to zero at the equilibrium point $p = q = 0$, $r = r_0$. Substituting $r = r_0 + x$ into eq. (17), we obtain

$$V = A(C - A)p^2 + B(C - B)q^2 + (A^2p^2 + B^2q^2 + C^2x^2 + 2Cr_0x)^2. \tag{18}$$

It is clear that the function (18) is positive definite for rather small x. The total time derivative of the function V has the form

$$\dot{V} = -2\varepsilon S\left\{\left[(C - A)p^2 + (C - B)q^2\right] + \left[A^2p^2 + B^2q^2 + C^2(x^2 + 2r_0x)\right](Ap^2 + Bq^2)\right\} \le 0. \tag{19}$$

So, the equilibrium point $p = q = 0$, $r = r_0$ is stable. Equations (12), (14), (16) and the phase portrait, showed in Fig. 2, give complete solution to the problem of spacecraft rotation evolution with an electric propulsion thruster.

4. Acknowledgment.

This work was supported by FAPESP (Process no.1996/3247-0)/INPE, from São Paulo state, Brazil.

5. References

[1] Grammel,R.: Der selbsterregte unsymmetrishe Kreisel. Ing.-Arch. **22**, 2, 73-97 (1954).

[2] Magnus, K.: Kreisel. Theorie und Anvendungen. Berlin–Heidelberg–New York: Springer 1971.

[3] Beletsky, V.V.: Asymptotic methods in rigid body dynamics. In: Problems of the Asymptotic Theory of Nonlinear Oscillations, pp. 42-46, Kiev, Nauk. dumka, 1977 (in Russian).

[4] Neishtadt, A.I., Pivovarov, M.L.: About evolution of rotation of spacecraft acted upon by a perturbing torque which is constant in a body frame. Space Research Institute, Moscow, Preprint N 434 (1978), (in Russian).

[5] Neishtadt, A.I.: Evolution of rotation of a rigid body acted upon by a sum of constant and dissipative perturbing moments. Mechanics of Solids **15**, 6, 30-36 (1980), (translated from Russian).

[6] Pivovarov, M.L.: The motion of a gyroscope with small self-excitation. Mechanics of Solids **20**, 6, 23-27 (1985), (translated from Russian).

[7] Chernous'ko, F.L.: The motion of a rigid body containing cavities filled with a viscous liquid with small Reynolds numbers. J. of Computational Mathematics and Mathematical Physics **5**, 6, 1049-1070, (1965), (translated from Russian).

[8] Chernous'ko, F.L.: Asymptotic methods in dynamics of systems with elastic and dissipative elements. In: Boundary and Interior Layers (Godunov, S.K, Miller, J.J.H., Novikov, V.A., eds.) pp. 15-26, 4th Int. BAIL-IV Conference, Novosibirsk, Russia 1986. Doublin: Boole Press 1986.

[9] Chernous'ko, F.L.: Asymptotic analysis for dynamics of a rigid body containing elastic elements and a viscous fluid. In: Dynamical Problems of Rigid-Elastic Systems and Structures (Banichuk, N.V., Klimov, D.M., Schiehlen, W., eds.) pp. 55-64, IUTAM Symposium, Moscow, Russia, 1990. Berlin: Springer 1991.

[10] Leimanis, E.: The General Problem of the Motion of Coupled Rigid Bodies About a Fixed Point. Berlin–Heidelberg–New York: Springer 1965.

[11] Sanders, J.A., Verhulst, F.: Averaging Methods in Nonlinear Dynamical Systems. New York: Springer 1985.

[12] Byrd, P.F., Friedman, M.D.: Handbook of Elliptic Integrals for Engineers and Scientists. Berlin: Springer 1971.

[13] Meirovitch, L.: Methods of Analytical Dynamics. New York: McGraw-Hill 1970.

Authors' addresses: M.L.Pivovarov, Space Research Institute, 117810 Profsoyuznaya 84/32, Moscow, Russia, e-mail: mpivovar@vm1.iki.rssi.ru; L.O.Ferreira and R.V.F.Lopes, INPE/DMC, Av. dos Astronautas, 1758 - Cx. Postal 515 - 12.201-970 S.J.Campos, Brazil.

Figure 1: The problem of Grammel
Figure 2: Rotation jet damping

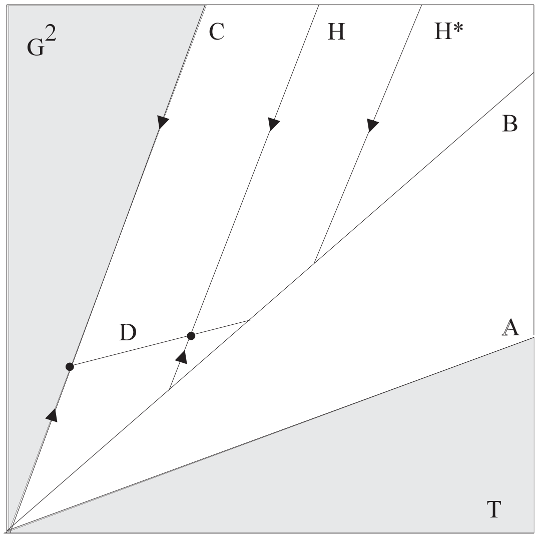

Figure 1: The problem of Grammel

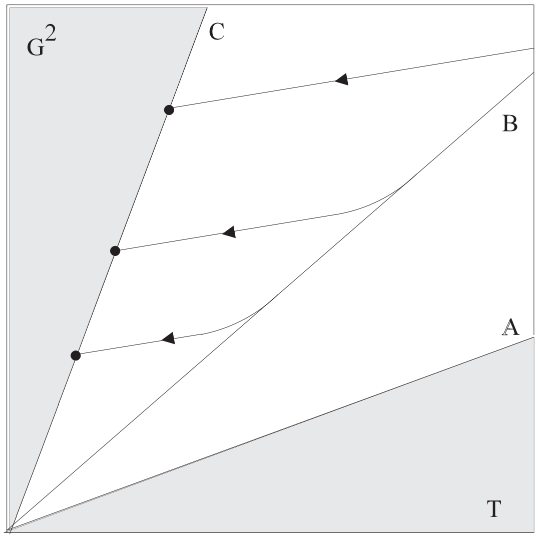

Figure 2: Rotation jet damping